Angular Momentum in Quantum Mechanics

In classical mechanics, the angular momentum

$$\mathbf{L} = \mathbf{r} \times \mathbf{p}$$

of any particle moving in a central field of force is conserved. For the reduced two-body problem, this is the content of *Kepler's Second Law*. Angular momentum also plays a crucial role in Quantum Mechanics. By "quantizing" the classical angular momentum above (i.e. setting $\mathbf{p} = -i\hbar\nabla$), we get the Orbital Angular Momentum Operator

$$\mathbf{L} = \mathbf{r} \times \frac{\hbar}{i}\nabla$$

whose three components

$$\begin{aligned} L_x &= \frac{\hbar}{i}(y\partial_z - z\partial_y) \\ L_y &= \frac{\hbar}{i}(z\partial_x - x\partial_z) \\ L_z &= \frac{\hbar}{i}(x\partial_y - y\partial_x) \end{aligned}$$

are well-defined self-adjoint operators [1] on $H^1(\mathbb{R}^3)$. Standard rules for computing commutators along with the fundamental commutation relations

$$[x_i, p_{x_j}] = i\hbar\delta_{ij}$$

show that:

$$[L_x, L_y] = i\hbar L_z \;\; [L_y, L_z] = i\hbar L_x \;\; [L_z, L_x] = i\hbar L_y.$$

In addition, the operator $L^2 = L_x^2 + L_y^2 + L_z^2$ commutes with all three components of $\mathbf{L}$. Hence, we can only have simultaneous eigenfunctions for L^2 and a single component of $\mathbf{L}$ (typically chosen to be L_z). As it turns out, many of the properties of angular momentum can be deduced simply from these algebraic relations! As such, we will study generic operators satisfying the commutation relations above. The following is taken primarily from [**?**, Chpts. 11 and 17] with a few examples drawn from [**?**]. Throughout, we use Dirac's bra|ket notation.

[1] Physics texts often use the (slightly misleading) term *Hermitian* for self-adjoint.

Generic Angular Momenta

Definition. *A (generic) angular momentum* $\mathbf{J}$ *is a triple of Hermitian operators* J_x, J_y *and* J_z *(on some appropriate Hilbert space* [2]*) satisfying the following commutation relations:*

$$[J_x, J_y] = i\hbar J_z \quad ; \quad [J_y, J_z] = i\hbar J_x \quad ; \quad [J_z, J_x] = i\hbar J_y \tag{1}$$

For any such triple of operators, we also define $J^2 \equiv J_x^2 + J_y^2 + J_z^2$ *together with the operators* $J_+ \equiv J_x + iJ_y$ *and* $J_- \equiv J_x - iJ_y$ *(known as raising and lowering operators, respectively).*

Note that although J_+ and J_- are not self-adjoint, they are adjoints of one another:

$$J_+^\dagger = J_- \text{ and } J_-^\dagger = J_+.$$

The following relations are simple consequences of the commutation relations (1):

$$\begin{aligned} [J_z, J_+] &= \hbar J_+ \\ [J_-, J_z] &= \hbar J_- \\ [J_+, J_-] &= 2\hbar J_z \\ [J^2, \mathbf{J}] &= 0 \end{aligned}$$

where the last relation is understood component-wise. There are also the very useful identities

$$J^2 - J_z^2 \pm \hbar J_z = J_\pm J_\mp$$

$$J^2 - J_z^2 = J_x^2 + J_y^2 = \frac{1}{2}(J_+J_- + J_-J_+) = \frac{1}{2}(J_+J_+^\dagger + J_+^\dagger J_+).$$

As an aside, note that the first three of the commutation relations above are precisely the ones for the Lie algebra $\mathfrak{sl}(2)$. As such, the eigenfunctions we discuss below constitute a representation of this Lie algebra, and most of what follows is a direct consequence of this fact!

Since J^2 and J_z commute, we can look for simultaneous eigenfunctions for these operators. We use the notation $|\lambda, m\rangle$, where these functions are assumed to be normalized and

$$\begin{aligned} J_z|\lambda, m\rangle &= m\hbar\,|\lambda, m\rangle \\ J^2|\lambda, m\rangle &= \lambda\hbar^2\,|\lambda, m\rangle \end{aligned}$$

[2] Often referred to as the "state space" in orthodox quantum theory.

(the use of $\hbar$ above is simply a convenient device to make the parameters m and λ physically dimensionless [3]). The first important fact to note is that

$$\lambda \geq m^2.$$

To see this, note that any operator of the form $AA^\dagger$ must have non-negative eigenvalues. Since $J^2 - J_z^2$ is the sum of two such operators, we get the result above.

Next, we consider the action of the operators J_+ and J_- on $|\lambda, m\rangle$ (via the commutation relations above).

$$\begin{aligned} J_z J_+|\lambda, m\rangle &= J_+ J_z|\lambda, m\rangle + \hbar\, J_+|\lambda, m\rangle &&= (m+1)\hbar\, J_+|\lambda, m\rangle \\ J_z J_-|\lambda, m\rangle &= J_- J_z|\lambda, m\rangle - \hbar\, J_-|\lambda, m\rangle &&= (m-1)\hbar\, J_-|\lambda, m\rangle \end{aligned}$$

$$J^2 J_\pm|\lambda, m\rangle \;\; = J_\pm J^2|\lambda, m\rangle \;\; = \lambda\hbar^2\, J_\pm|\lambda, m\rangle$$

From these relations (and the orthogonality of eigenfunctions corresponding to different eigenvalues) we conclude that

$$\begin{aligned} J_+|\lambda, m\rangle &= C_+(\lambda, m)\hbar|\lambda, m+1\rangle \\ J_-|\lambda, m\rangle &= C_-(\lambda, m)\hbar|\lambda, m-1\rangle \end{aligned}$$

where the complex coefficients $C_+(\lambda, m)$ and $C_-(\lambda, m)$ will be determined below.

Our observation that $\lambda \geq m^2$ limits the number of times we can apply the raising and lowering operators for a given value of λ. Let the maximum value of m be j and the minimum value be j'. Hence

$$J_+|\lambda, j\rangle = 0 \text{ and } J_-|\lambda, j'\rangle = 0.$$

We now observe the following:

$$\begin{aligned} J_- J_+|\lambda, j\rangle &= (J^2 - J_z^2 - \hbar J_z)|\lambda, j\rangle \\ &= (\lambda - j^2 - j)\hbar^2\,|\lambda, j\rangle \end{aligned}$$

from which we conclude $\lambda = j(j+1)$. Similarly, considering $J_+J_-|\lambda, j'\rangle$ gives $\lambda = j'(j'+1)$. Since we must have $j \geq j'$ by assumption, the only way these two equalities can be true is if $j' = -j$. As such, we relabel our eigenfunctions by $|j, m\rangle$ where

$$\begin{aligned} J_z|j, m\rangle &= m\hbar\,|j, m\rangle \\ J^2|j, m\rangle &= j(j+1)\hbar^2\,|j, m\rangle \end{aligned}$$

[3] We could have also introduced the operator $\hat{J} = \frac{1}{\hbar}J$ to accomplish this.

Starting from the lowest value of $m = -j$, it must be possible to get to the highest value, j, by repeated applications of the raising operator. Hence, $2j$ must be a non-negative integer. Thus

$$j \in \mathbb{Z}_{\geq 0} \cup \left(\mathbb{Z}_{\geq 0} + \frac{1}{2} \right).$$

As a result, for every possible j we get a collection of $2j + 1$ eigenfunctions (i.e. one for each possible value of $m \in \{-j, -j+1, \ldots, j-1, j\}$).

Picking up on the last aside, the representation theory of $\mathfrak{sl}(2)$ shows that the irreducible finite-dimensional representations of this Lie algebra are in one-to-one correspondence with the non-negative integers. Hence, for each fixed value of j, we simply get the $(2j+1)$-dimensional representation of $\mathfrak{sl}(2)$!

Finally, we compute the coefficients appearing in the action of the raising and lowering operators. Recalling that $J_- = J_+^\dagger$ gives us the following

$$\langle j, m | J_- J_+ | j, m \rangle = \left(\langle j, m | J_+^\dagger \right) J_+ | j, m \rangle = |C_+(j, m)|^2 \hbar^2.$$

On the other hand

$$\begin{aligned} \langle j, m | J_- J_+ | j, m \rangle &= \langle j, m | J^2 - J_z^2 - \hbar J_z | j, m \rangle \\ &= (j(j+1) - m^2 - m) \hbar^2 \end{aligned}$$

and so

$$\begin{aligned} |C_+(j, m)|^2 &= j(j+1) - m^2 - m \\ &= (j - m)(j + m + 1). \end{aligned}$$

Since the phase of the coefficient is undetermined, we choose it to be zero. This and a similar calculation with $J_+ J_-$ gives

$$\begin{aligned} J_+ | j, m \rangle &= \sqrt{(j - m)(j + m + 1)}\, \hbar \, | j, m + 1 \rangle \\ J_- | j, m \rangle &= \sqrt{(j + m)(j - m + 1)}\, \hbar \, | j, m - 1 \rangle \end{aligned}$$

Note that commutation relations (1) have completely determined the spectrum of eigenvalues for ANY (generic) angular momentum! Of course, the determination of the corresponding eigenfunctions depends on the exact nature of **J**. As a word of warning, not every angular momentum operator admits every possible j in $\mathbb{Z}_{\geq 0} \cup \left(\mathbb{Z}_{\geq 0} + \frac{1}{2} \right)$. In particular, the orbital angular momentum operator **L** only admits $j \in \mathbb{Z}_{\geq 0}$ (this is because the eigenfunctions involve $e^{im\phi}$ which forces m to be integral as the azimuthal angle ϕ is 2π periodic - c.f. [**?**, pp. 242 – 244]).

Addition of Angular Momenta

Suppose that we have two distinct angular momenta $\mathbf{J}_1$ and $\mathbf{J}_2$ which act on distinct spaces H_1 and H_2, respectively. It is natural to form the operator

$$\mathbf{J} = \mathbf{J}_1 \otimes 1 + 1 \otimes \mathbf{J}_2$$

acting on the tensor product space $H_1 \otimes H_2$. The eigenvectors of J_i^2 and J_{zi} give a basis of H_i and so the tensor product of these elements gives a basis of $H_1 \otimes H_2$. We will use the notation

$$|j_1, j_2, m_1, m_2\rangle \equiv |j_1, m_1\rangle \otimes |j_2, m_2\rangle$$

and simplify the notation for the operators by suppressing tensor products with the identity.

The first thing to check is that $\mathbf{J}$ satisfies the commutation relations for angular momentum. This is trivial since the operators $\mathbf{J}_1$ and $\mathbf{J}_2$ commute on our state space. The operator $J_z \equiv J_{z1} + J_{z2}$ commutes with J_1^2, J_2^2, J_{z1} and J_{z2}, but J^2 **does not** commute with either J_{1z} or J_{2z}. We wish to trade our set of commuting operators $\{J_1^2, J_{z1}, J_2^2, J_{2z}\}$ for the set $\{J_1^2, J_2^2, J^2, J_z\}$. Denote the eigenfunctions of this second set by $|j_1, j_2, j, m\rangle$. Since our original collection of eigenfunctions constituted a basis, we must have

$$|j_1, j_2, j, m\rangle = \sum_{m_1, m_2} |j_1, j_2, m_1, m_2\rangle\langle j_1, j_2, m_1, m_2|j_1, j_2, j, m\rangle.$$

The summation is only over the indices m_1 and m_2 since the operators J_1^2 and J_2^2 are common to both sets of observables (which forces the corresponding eigenvalues to be equal). The problem here is to determine the coefficients $\langle j_1, j_2, m_1, m_2|j_1, j_2, j, m\rangle$ - known as *Clebsch-Gordon* or *Wigner* coefficients (we refer to them as C-G coefficients below).

In the calculations that follow, we will often be thinking of the eigenvalues j_1 and j_2 as fixed. So, in order to make the notation more manageable we will write

$$\begin{aligned} |m_1, m_2\rangle &\equiv |j_1, j_2, m_1, m_2\rangle \\ |j, m\rangle &\equiv |j_1, j_2, j, m\rangle \\ \langle m_1, m_2|j, m\rangle &\equiv \langle j_1, j_2, m_1, m_2|j_1, j_2, j, m\rangle. \end{aligned}$$

To start the process, first note that applying the operator J_z to the expansion above gives

$$m\hbar\,|j, m\rangle = \sum_{m_1, m_2} (m_1 + m_2)\hbar\,|m_1, m_2\rangle\langle m_1, m_2|j, m\rangle.$$

This forces

$$\langle m_1, m_2|j, m\rangle = 0 \text{ if } m_1 + m_2 \neq m$$

which is known as the *selection rule* for C-G coefficients.

Next we develop powerful *recursion formulae* for these coefficients. Applying $J_- = J_{1-} + J_{2-}$ to the expansion above gives

$$\begin{aligned}\sqrt{(j+m)(j-m+1)}\,\hbar\,|j,m-1\rangle &= \sum_{n_1,n_2} \hbar\,\langle n_1,n_2|j,m\rangle \\ &\quad \cdot\{\sqrt{(j_1+n_1)(j_1-n_1+1)}\,|n_1-1,n_2\rangle \\ &\quad +\sqrt{(j_2+n_2)(j_2-n_2+1)}\,|n_1,n_2-1\rangle\}.\end{aligned}$$

Taking inner products with $\langle m_1,m_2|$ and appealing to the orthonormality of our eigenfunctions gives

$$\begin{aligned}\sqrt{(j+m)(j-m+1)}\,\langle m_1,m_2|j,m-1\rangle &= \sqrt{(j_1+m_1+1)(j_1-m_1)}\,\langle m_1+1,m_2|j,m\rangle \\ &\quad +\sqrt{(j_2+m_2+1)(j_2-m_2)}\,\langle m_1,m_2+1|j,m\rangle.\end{aligned}$$

Performing a similar calculation using J_+ gives a similar relation. We summarize both below:

$$\begin{aligned}\sqrt{(j\pm m)(j\mp m+1)}\,\langle m_1,m_2|j,m\mp 1\rangle &= \sqrt{(j_1\mp m_1)(j_1\pm m_1+1)}\,\langle m_1\pm 1,m_2|j,m\rangle \\ &\quad +\sqrt{(j_2\mp m_2)(j_2\pm m_2+1)}\,\langle m_1,m_2\pm 1|j,m\rangle.\end{aligned}$$

The power of these recursion relations is that all the C-G coefficients corresponding to fixed values of j_1, j_2 and j can be completely determined once we know one of them; for instance, knowing the value of $\langle j_1, j-j_1|j,j\rangle$ gives us all the others corresponding to the same values noted above. This one coefficient must be chosen to normalize the expansion of $|j,m\rangle$. This still leaves the phase undetermined. Typically, $\langle j_1, j-j_1|j,j\rangle$ is chosen to be positive. If this is done, then all the C-G coefficients are real. We will illustrate this procedure below.

Since $j-j_1$ is an eigenvalue of J_{2z} in $\langle j_1, j-j_1|j,j\rangle$, we must have $j_1-j_2 \le j \le j_1+j_2$ if this coefficient is to be non-zero. However, since we could have just as easily used the coefficient $\langle j-j_2, j_2|j,j\rangle$ to generate the C-G coefficients, we must also have $j_2-j_1 \le j \le j_1+j_2$. This leads to the following *triangular condition*

$$|j_1-j_2| \le j \le j_1+j_2$$

for j_1, j_2 and j. Despite appearances, the three eigenvalues j_1, j_2 and j are on an equal footing:

$$|j-j_1| \le j_2 \le j+j_1$$

$$|j-j_2| \le j_1 \le j+j_2$$

(which can be shown by analogous arguments as the original). Finally, note that since $m = m_1 + m_2$ ranges between $-j$ and j in steps of one, we must have $j = |j_1 - j_2|, |j_1 - j_2| + 1, \ldots, j_1 + j_2 - 1, j_1 + j_2$. This forces all three to be integers or two to be half-integers and the other an integer.

Finally, we note three more important properties of C-G coefficients. The first two are direct consequences of our eigenfunctions being orthonormal and choosing the C-G coefficients real:

$$\sum_{m_1,m_2} \langle m_1, m_2 | j, m \rangle \langle m_1, m_2 | j', m' \rangle = \delta_{jj'} \delta_{mm'}$$

$$\sum_{j,m} \langle m_1, m_2 | j, m \rangle \langle m_1', m_2' | j, m \rangle = \delta_{m_1 m_1'} \delta_{m_2 m_2'}$$

The last property involves switching the values of j_1 and j_2 (so we reinstate the full notation):

$$\begin{aligned} \langle j_1, j_2, m_1, m_2 | j_1, j_2, j, m \rangle &= (-1)^{j-j_1-j_2} \langle j_2, j_1, m_2, m_1 | j_2, j_1, j, m \rangle \\ &= \langle j_2, j_1, -m_2, -m_1 | j_2, j_1, j, -m \rangle. \end{aligned}$$

Since we will not need this last symmetry relation, we refer the curious reader to [**?**, pp. 1040 - 1042].

A Sample Calculation

We will consider the total angular momentum of a spin $\frac{1}{2}$ particle. That is, we have two angular momenta **S** and **L** (the orbital angular momentum operator) where $j_S = \frac{1}{2}$. To be definite, we take $j_L = 2$. We wish to find the simultaneous eigenvectors for the operators $\{S^2, L^2, J^2, J_z\}$ where $\mathbf{J} = \mathbf{S} + \mathbf{L}$ is the total angular momentum operator. The state space in this case is spanned by the following simultaneous eigenfunctions of $\{S^2, S_z, L^2, L_z\}$ (where we suppress $j_S = \frac{1}{2}$ and $j_L = 2$)

$$\left\{ \left| -\frac{1}{2}, -2 \right\rangle_0, \left| -\frac{1}{2}, -1 \right\rangle_0, \left| -\frac{1}{2}, 0 \right\rangle_0, \left| -\frac{1}{2}, 1 \right\rangle_0, \left| -\frac{1}{2}, 2 \right\rangle_0, \left| \frac{1}{2}, -2 \right\rangle_0, \left| \frac{1}{2}, -1 \right\rangle_0, \left| \frac{1}{2}, 0 \right\rangle_0, \left| \frac{1}{2}, 1 \right\rangle_0, \left| \frac{1}{2}, 2 \right\rangle_0 \right\}^4$$

By the triangular condition above, we know that the admissible values for j are $\{\frac{3}{2}, \frac{5}{2}\}$. So, the state space can also be spanned by the eigenfunctions of $\{S^2, L^2, J^2, J_z\}$:

$$\left\{ \left| \frac{3}{2}, -\frac{3}{2} \right\rangle_1, \left| \frac{3}{2}, -\frac{1}{2} \right\rangle_1, \left| \frac{3}{2}, \frac{1}{2} \right\rangle_1, \left| \frac{3}{2}, \frac{3}{2} \right\rangle_1, \left| \frac{5}{2_1}, -\frac{5}{2} \right\rangle_1, \left| \frac{5}{2}, -\frac{3}{2} \right\rangle_1, \left| \frac{5}{2}, -\frac{1}{2} \right\rangle_1, \left| \frac{5}{2}, \frac{1}{2} \right\rangle_1, \left| \frac{5}{2}, \frac{3}{2} \right\rangle_1, \left| \frac{5}{2}, \frac{5}{2} \right\rangle_1 \right\}$$

[4] The use of the subscript "0" (and "1" in the second set) merely helps distinguish the different bases.

(where again we have suppressed the fixed eigenvalues for S^2 and L^2). In calculating the expansions below, we freely apply the selection rule $m_S + m_L = m$ without further comment. The easy consequences of this are

$$\left|\frac{5}{2}, -\frac{5}{2}\right\rangle_1 = \left|-\frac{1}{2}, -2\right\rangle_0 \implies \left\langle -\frac{1}{2}, -2\right|\left.\frac{5}{2}, -\frac{5}{2}\right\rangle = 1$$
$$\left|\frac{5}{2}, \frac{5}{2}\right\rangle_1 = \left|\frac{1}{2}, 2\right\rangle_0 \implies \left\langle \frac{1}{2}, 2\right|\left.\frac{5}{2}, \frac{5}{2}\right\rangle = 1$$

We next determine the expansion of $\left|\frac{3}{2}, \frac{3}{2}\right\rangle_1$ which must be

$$\left|\frac{3}{2}, \frac{3}{2}\right\rangle_1 = \alpha\left|\frac{1}{2}, 1\right\rangle_0 + \beta\left|-\frac{1}{2}, 2\right\rangle_0$$

where $\alpha = \left\langle \frac{1}{2}, 1\right|\left.\frac{3}{2}, \frac{3}{2}\right\rangle$ and $\beta = \left\langle -\frac{1}{2}, 2\right|\left.\frac{3}{2}, \frac{3}{2}\right\rangle$. Using the lower signs in the recursion relations with $m_1 = \frac{1}{2}, m_2 = 2, j = \frac{3}{2}$ and $m = \frac{3}{2}$ gives:

$$\begin{aligned} 0 &= \left\langle -\frac{1}{2}, 2\right|\left.\frac{3}{2}, \frac{3}{2}\right\rangle + 2\left\langle \frac{1}{2}, 1\right|\left.\frac{3}{2}, \frac{3}{2}\right\rangle \\ &\text{or} \\ \beta &= -2\alpha \end{aligned}$$

and so we have

$$\left|\frac{3}{2}, \frac{3}{2}\right\rangle_1 = \alpha\left(\left|\frac{1}{2}, 1\right\rangle_0 - 2\left|-\frac{1}{2}, 2\right\rangle_0\right).$$

So, we should choose $\alpha = \frac{1}{\sqrt{5}}$ to normalize our eigenfunction. Hence, we find

$$\left\langle \frac{1}{2}, 1\right|\left.\frac{3}{2}, \frac{3}{2}\right\rangle = \frac{1}{\sqrt{5}} \quad \text{and} \quad \left\langle -\frac{1}{2}, 2\right|\left.\frac{3}{2}, \frac{3}{2}\right\rangle = -\frac{2}{\sqrt{5}}.$$

Note that we chose to make $\left\langle \frac{1}{2}, 1\right|\left.\frac{3}{2}, \frac{3}{2}\right\rangle$ based on the convention above that $\langle j_1, j - j_1 | j, j\rangle$ always be chosen real and positive. Of course, this depends on how you order the operators! In tables of C-G coefficients, the order is typically determined by maximum possible value of j when these are different for the two operators. So in most tables, the signs given above would be reversed (but the magnitude of the coefficients will be the same). The remaining C-G coefficients for this (and other cases) are listed in the attached chart (bearing in mind the caveat about sign conventions).

References

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