Peetre's Theorem

On a C^∞ manifold, one can define the notion of a differential operator on C^∞-functions by appealing to the natural definition on open subsets of $\mathbb{R}^n$ and using local coordinates to give this meaning on the manifold. While such a definition is useful for calculations, it seems a bit clumsy and inelegant. The theorem by Peetre proven below gives a simple characterization of differential operators on manifolds without appealing to local coordinates. The proof below is adapted from the proof in [He84, pp. 233–239] which itself is adapted from the proof given in [Na68, pp. 171–176]. We begin with some standard notation and propositions before coming to the main course of Peetre's Theorem.

Notation and Basic Propositions

Let $V \subset \mathbb{R}^n$ be open. Denote by $C_c^\infty(V)$ the set of functions compactly supported inside V. Let ∂_i denote the partial derivative with respect to the i-th coordinate in $\mathbb{R}^n$, and for a multi-index $\alpha = (\alpha_1, \ldots, \alpha_n)$ of non-negative integers we put

$$\begin{array}{rcl} D^\alpha = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n} & , & x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n} \\ |\alpha| = \alpha_1 + \cdots + \alpha_n & , & \alpha! = \alpha_1! \cdots \alpha_n! \end{array}$$

In addition, if $\beta \leq \alpha$ (i.e. $\beta_i \leq \alpha_i$ for every i) then we can also define

$$\begin{aligned} \alpha - \beta &= (\alpha_1 - \beta_1, \ldots, \alpha_n - \beta_n) \\ \begin{pmatrix} \alpha \\ \beta \end{pmatrix} &= \frac{\alpha!}{\beta!(\alpha-\beta)!}. \end{aligned}$$

With this very convenient notation, the generalized Leibniz rule for the differentiation of two (sufficiently differentiable) functions f and g can be written as

$$D^\alpha(fg) = \sum_{\mu+\nu=\alpha} \begin{pmatrix} \alpha \\ \mu \end{pmatrix} D^\mu(f) D^\nu(g).$$

For any subset $S \subset V$ and $m \in \mathbb{Z}_{\geq 0}$ we define

$$\|f\|_m^S = \sum_{|\alpha| \leq m} \sup_{x \in S} |(D^\alpha f)(x)|.$$

When $S = V$ we will drop the superscript. Clearly, these form a countable family of semi-norms on $C^\infty(V)$.

Proposition 1. *Let $C \subset U \subset V$ where U and V are open and C a closed subset of $\mathbb{R}^n$. Then there is a function $\phi \in C^\infty(V)$ such that $\phi \equiv 1$ on C, $\phi \equiv 0$ on $V \setminus U$, and $0 \leq \phi \leq 1$ everywhere on V.*

Proposition 2. *Let $C \subset \mathbb{R}^n$ be any closed subset. Then there exists a $\phi \in C^\infty(\mathbb{R}^n)$ such that*

$$C = \{x \in \mathbb{R}^n : \phi(x) = 0\}.$$

For the construction of such functions, see [Sp05, pp. 32–32] and [He84, pp. 234-235].

Lemma 1. *Let $m > 0$, and suppose $f \in C^\infty(\mathbb{R}^n)$ has all derivatives of order less than or equal to m vanish at the origin. Then given $\epsilon > 0$ there exists a $g \in C^\infty(\mathbb{R}^n)$ vanishing in a neighborhood of the origin and satisfying*

$$\|g - f\|_m < \epsilon.$$

Proof: Let $\phi \in C^\infty(\mathbb{R}^n)$ be such that $\phi(x) = 0$ for $|x| \leq \frac{1}{2}$, $\phi(x) = 1$ for $|x| \geq 1$, and $0 \leq \phi \leq 1$ everywhere. For $\delta > 0$ define

$$g_\delta(x) = \phi\left(\frac{x}{\delta}\right) f(x).$$

Note that $g_\delta \equiv 0$ on $|x| \leq \frac{\delta}{2}$ and $g_\delta \equiv f$ on $|x| \geq \delta$. Hence, the theorem will follow if we can show that for each multi-index $|\alpha| \leq m$

$$\sup_{|x| \leq \delta} |D^\alpha g_\delta(x) - D^\alpha f(x)| \to 0 \text{ as } \delta \to 0.$$

By the assumption that f and its derivatives up to order m vanish at the origin, we know that

$$\sup_{|x| \leq \delta} |D^\alpha f(x)| \to 0 \text{ as } \delta \to 0.$$

Using the Leibniz rule and the chain rule shows that

$$D^\alpha g_\delta(x) = \sum_{\mu+\nu=\alpha} \binom{\alpha}{\mu} \delta^{-|\mu|} (D^\mu \phi)\left(\frac{x}{\delta}\right) (D^\nu f)(x).$$

Since ϕ is a fixed function, bounded in absolute value by 1, and constant outside a compact set (so that all of its derivatives have compact support), we can get an upper bound on $(D^\mu\phi)(x)$. Thus, we can conlcude that there is a constant C_1 so that

$$|D^\alpha g_\delta(x)| \leq C_1 \sum_{\mu+\nu=\alpha} \delta^{-|\mu|}|(D^\nu f)(x)|.$$

But $D^\nu f$ is such that all of its derivatives up to order $m - |\nu|$ vanish at the origin. So, expanding $D^\nu f$ about the origin, we can show that

$$\sup_{|x|\leq\delta} |D^\nu f(x)| \leq C_2 \delta^{m+1-|\nu|}.$$

Hence, we conclude that

$$\sup_{|x|\leq\delta} |D^\alpha g_\delta(x)| \leq C_1 C_2 \sum_{\mu+\nu=\alpha} \delta^{m+1-|\nu|-|\mu|}.$$

Since $|\nu| + |\mu| = |\alpha| \leq m$ we see that

$$\sup_{|x|\leq\delta} |D^\alpha g_\delta(x)| \to 0 \text{ as } \delta \to 0. \square$$

Corollary. *The lemma above holds if the origin is replaced by any point $y \in \mathbb{R}^n$.*

Peetre's Theorem on $\mathbb{R}^n$

Defintion. *Let $V \subset \mathbb{R}^n$ be an open set. A differential operator on V is a linear mapping $L : C_c^\infty(V) \to C_c^\infty(V)$ such that for each open set U compactly contained in V there exists a finite family of functions $a_\alpha \in C^\infty(U)$ so that*

$$L\phi = \sum_\alpha a_\alpha D^\alpha \phi$$

for every $\phi \in C_c^\infty(U)$.

Proposition 3. *Let L be a differential operator on V, then we have*

$$supp(L\psi) \subset supp(\psi) \text{ for all } \psi \in C_c^\infty(V). \qquad (\dagger)$$

Peetre's Theorem proves the amazing result that (†) is also *sufficient* for a linear operator on $C_c^\infty(V)$ to be a differential operator! We first establish this on $\mathbb{R}^n$ and then extend the result to smooth manifolds.

Lemma 2. *Suppose ψ_1 and ψ_2 in $C_c^\infty(V)$ agree on some neighborhood of $x \in V$ and the linear operator L satisfies (†), then $L\psi_1(x) = L\psi_2(x)$.*

Proof: Clearly $x \notin supp(\psi_1 - \psi_2)$. So, by assumption $x \notin supp(L(\psi_1 - \psi_2))$ and so $L\psi_1(x) = L\psi_2(x)$ by linearity of L. □

Corollary. *If L is a linear operator defined on $C_c^\infty(V)$ satisfying (†), then L can be extended to all of $C^\infty(V)$.*

Proof: Let ϕ_x be any element of $C_c^\infty(V)$ that is identically 1 on a neighborhood of x. For any $f \in C^\infty(V)$ define

$$Lf(x) = L(\phi_x(x) \cdot f(x)).$$

By the lemma above, the particular choice of ϕ_x is irrelevant. □

The next two lemmas are the main technical results that facilitate the proof of the theorem.

Lemma 3. *Suppose $L : C_c^\infty(V) \to C_c^\infty(V)$ is linear and satisfies (†). Then every $a \in V$ has an open neighborhood U compactly contained in V together with a non-negative integer m and a positive constant C so that*

$$\|Lu\|_0 \leq C\|u\|_m \text{ for all } u \in C_c^\infty(U \setminus \{a\}).$$

Proof: Suppose this were false for some $a \in V$. Let $U_0 \subset V$ be an open neighborhood of a that is compactly contained in V. Then by assumption, there must be a function $u_1 \in C_c^\infty(U_0 \setminus \{a\})$ such that $\|Lu_1\|_0 > 2^2\|u_1\|_1$.

Let $U_1 = \{x : u_1(x) \neq 0\}$. Then $U_0 \setminus \bar{U_1}$ is an open neighborhood of a. Again by our assumption, there must be a $u_2 \in C_c^\infty(U_0 \setminus (\bar{U_1} \cup \{a\}))$ such that $\|Lu_2\|_0 > 2^4\|u_2\|_2$. Continue this process by induction to get a sequence of open sets $U_1, U_2, \ldots$ with

$$\bar{U_k} \subset U_0 \setminus \{a\}$$

$$\bar{U_k} \cap \bar{U_l} = \emptyset \text{ for } k \neq l$$

and a sequence of functions $u_k \in C_c^\infty(U_0 \setminus (\bar{U_1} \cup \cdots \cup \bar{U}_{k-1} \cup \{a\})) \subset C_c^\infty(V)$ satisfying

$$U_k = \{x : u_k(x) \neq 0\}$$
$$\|Lu_k\|_0 > 2^{2k}\|u_k\|_k.$$

Now define

$$u(x) = \sum_{i=1}^{\infty} \frac{2^{-k}}{\|u_i\|_i} u_i(x).$$

It is easy to show that this is a well-defined element of C_c^∞. Moreover, by construction

$$u|_{U_k} = 2^{-k}\|u_k\|^{-1}u_k$$

which by linearity and our support condition (†) implies that

$$Lu|_{U_k} = 2^{-k}\|u_k\|^{-1}Lu_k.$$

Since $\|Lu_k\|_0 > 2^{2k}\|u_k\|_k$, there must be $x_k \in U_k$ so that $Lu(x_k) > 2^k$. Hence, Lu is unbounded which is impossible since $Lu \in C_c^\infty(V)$ by assumption. □

Lemma 4. *Suppose $L : C_c^\infty(V) \to C_c^\infty(V)$ is linear and satisfies (†). Let $U \subset V$ be any open set, and assume there is a constant $C > 0$ and a non-negative integer m so that*

$$\|Lu\|_0 \leq C\|u\|_m \text{ for all } u \in C_c^\infty(U).$$

Then there are functions $a_\alpha \in C^\infty(V)$ such that

$$(Lu)(x) = \sum_{|\alpha| \leq m} a_\alpha(x)(D^\alpha u)(x)$$

for all $x \in U$ and $u \in C_c^\infty(U)$.

Proof: By the corollary to Lemma 2, we can extend L to all of $C^\infty(V)$. For each $a \in V$ and each multi-index α, define

$$\begin{aligned} Q_{\alpha,a}(x) &= (x_1 - a_1)^{\alpha_1} \cdots (x_n - a_n)^{\alpha_n} \\ b_\alpha(a) &= (LQ_{\alpha,a})(a). \end{aligned}$$

Clearly, the functions b_α belong to $C^\infty(V)$. For any $u \in C_c^\infty(U)$ and $a \in U$, consider the function $f_{a,m}$ defined by

$$f_{a,m} = u - \sum_{|\alpha| \leq m} \frac{(D^\alpha u)(a)}{\alpha!} Q_{\alpha,a} \tag{1}$$

(note that $f_{a,m}$ is just the difference between u and its Taylor Polynomial of order m centered at a). By construction, $f_{a,m}$ has all derivatives or order $|\alpha| \leq m$ vanishing at a. By the proof of Lemma 1, we can approximate $f_{a,m}$ in the semi-norm $\|\cdot\|_m$ by functions g_ν which exactly equal $f_{a,m}$ outside some neighborhood of a but vanish identically near a. By (†) and the corollary to Lemma 2, we must have Lg_ν vanish identically near a for all ν. Thus, by our assumption

$$\|L(f_{a,m} - g_\nu)\|_0 \leq \|f_{a,m} - g_\nu\|_m \to 0 \text{ as } \nu \to \infty$$

and so

$$(Lf_{a,m})(a) = \lim_{\nu\to\infty}(Lg_\nu)(a) = 0.$$

Applying L to both sides of (1) and using $(Lf_{a,m})(a) = 0$ gives

$$(Lu)(a) = \sum_{|\alpha|\leq m} \frac{(LQ_{\alpha,a})(a)}{\alpha!}(D^\alpha u)(a).$$

Taking $a_\alpha(x) = \frac{b_\alpha(x)}{\alpha!}$ we get the desired result. □

We finally have all the necessary technical elements to prove

Theorem 1 (Peetre's Theorem). *Suppose $L : C_c^\infty(V) \to C_c^\infty(V)$ is linear and satisfies supp$(L\psi) \subset$ supp(ψ) for all $\psi \in C_c^\infty(V)$. Then L is a differential operator.*

Proof: Let $U \subset V$ be open with $\bar{U}$ compact and $\bar{U} \subset V$. Applying Lemma 3 to every point of $\bar{U}$ gives us an open cover. By the compactness of $\bar{U}$ we get a finite cover of open sets $U_1, \dots, U_r$ neighborhoods of points $a_1, \dots, a_r$ in $\bar{U}$, respectively; by the construction in the lemma, we can take each $\bar{U}_i$ to be compact. For each i, we know that there are constants $C_i > 0$ and positive integers m_i so that $\|Lu\|_0 \leq C_i\|u\|_{m_i}$ for every $u \in C_c^\infty(U_i \setminus \{a_i\})$ Set $C = \max\{C_i\}$ and $m = \max\{m_i\}$. Then we have

$$\|Lu\|_0 \;\leq\; C\|u\|_m \tag{2}$$

for each i and every $u \in C_c^\infty(U_i \setminus \{a_i\})$. Let $1 = \sum_{i=1}^{r+1} \phi_i$ be a C^∞-partition-of-unity subordinate to the covering $U_1, \dots, U_r, V \setminus \bar{U}$ of V (c.f. [Sp05, pp. 51 – 52]).

Let $u \in C_c^\infty(U \setminus \{a_1, \ldots, a_r\})$, then we can write $u = \sum_{i=1}^{r+1} \phi_i u$ and (2) holds for each $\phi_i u$. Hence, we have

$$\|Lu\|_0 = \|\sum_{i=1}^{r} L(\phi_i u)\|_0 \leq \sum_{i=1}^{r} \|L(\phi_i u)\|_0 \leq C \sum_{i=1}^{r} \|\phi_i u\|_m.$$

Using the Leibniz rule to expand the derivatives of $\phi_i u$ and realizing that the ϕ_i are fixed functions of compact support allows us to conclude that for any $u \in C_c^\infty(U \setminus \{a_1, \ldots, a_r\})$

$$\|Lu\|_0 \leq C' \|u\|_m$$

for some (possibly different) fixed constant C'.

By Lemma 4, we can conclude that

$$(Lu)(x) = \sum_{|\alpha| \leq m} a_\alpha(x)(D^\alpha u)(x)$$

for all $x \in U \setminus \{a_1, \ldots, a_r\}$ and some collection of functions $a_\alpha \in C^\infty(V)$. But, both sides are continuous! Hence they must be equal on all of U. □

Peetre's Theorem and Differential Operators on Manifolds

As is typical, we can use the local result in $\mathbb{R}^n$ to prove a similar result on smooth manifolds. To that end, we formulate the following (non-standard) definition.

Defintion. *Let M be a C^∞-manifold. A differential operator on M is a linear operator $L : C_c^\infty(M) \to C_c^\infty(M)$ satisfying supp$(L\psi) \subset$ supp(ψ) for all $\psi \in C_c^\infty(M)$*

The great advantage of this formulation is that it makes no reference to local coordinates. We now show that this definition is justified.

Theorem 2. *The definition above is equivalent to the usual definition of differential operators on manifolds.*

Proof: Let (U, ϕ) be a chart on M. Consider the mapping $L^\phi : C_c^\infty(\phi(U)) \to C_c^\infty(\phi(U))$ given by

$$L^\phi F = (L(F \circ \phi)) \circ \phi^{-1}.$$

An easy check shows that this satisfies the hypotheses of Peetre's Theorem. Hence, for any open set compactly contained in $\phi(U)$ we have

$$(L^{\phi}F)(x) = \sum_{\alpha} a_{\alpha}(x)(D^{\alpha}F)(x).$$

Thus, setting $F = f \circ \phi^{-1}$ where $f \in C_c^{\infty}(U)$ and $y = \phi^{-1}(x) \in U$ gives

$$Lf(y) = \sum_{\alpha} a_{\alpha}(\phi(y))(D^{\alpha}(f \circ \phi^{-1}))(\phi(y))$$

for every open W compactly contained in U. This is just the usual definition of a differential operator on a smooth manifold. That the usual definition implies our (non-standard) definition is trivial! □

References

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