Studies of the Relativistic Vlasov-Poisson System

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The *relativistic Vlasov-Poisson* (rVP) system (in units where $c = 1$) reads:

$$\mathrm{rVP}^{\pm} : \begin{cases} \left(\partial_t + \frac{p}{\sqrt{1+|p|^2}} \cdot \nabla_q \pm \nabla_q \varphi_t(q) \cdot \nabla_p\right) f_t(p,q) = 0 \\ \triangle_q \varphi_t(q) = 4\pi \int f_t(p,q)\, d^3p \\ \varphi_t(q) \asymp -|q|^{-1} \text{ as } |q| \to \infty. \end{cases}$$

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2001: Glassey and Schaeffer publish additional results on the global existence of symmetric solutions to rVP.

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2008: Lemou, Méhats, and Raphaël study non-linear stability versus the formation of singularities in rVP^- through concentration compactness techniques. They also show that systems launched by initial data with negative total energy approach self-similar collapse.

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- They find the sharp value of $\mathcal{C}_{3/2}^-$, but leave the remaining non-trivial constants as the solution of a minimization problem.
- They also show that *zero total energy* initial data with total virial less than or equal to $-1/2$ will blow up in finite time. However, they give no explicit examples of such data.
- Finally, the appendix to their paper presents a novel proposal whereby rVP$^-$ might obtain as the Vlasov Limit of an overall neutral two-specie charged plasma on certain space-time scales.

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2) Examples of initial data with zero total energy and virial less than or equal to $-1/2$ will be found. This answers in the affirmative the question of their existence by S. Calogero (private communication to S. Tahvildar-Zadeh).
3) We examine the proposal for the emergence of rVP$^-$ from overall neutral two-specie plasma interacting via *Coulombic* forces.

Optimal $\mathfrak{L}^{\beta}$-Control for the Global Cauchy Problem for rVP^{-}

The critical constant $\mathcal{C}_{\beta}^{-}$ is given by the following minimization problem:

$$\mathcal{C}_{\beta}^{-} \equiv \inf_{\mathfrak{P}_1 \cap \mathfrak{L}^{\beta}} \Phi_{\beta}(f),$$

$$\Phi_{\beta}(f) \equiv \left(\frac{\mathcal{E}_p^u(f)}{-\mathcal{E}_q(f)} \right)^{3(1-\frac{1}{\beta})} \|f\|_{\beta},$$

The functionals $\mathcal{E}_p^u$ and $\mathcal{E}_q$ are given by

$$\mathcal{E}_p^u(f) \equiv \iint |p| f(p,q)\, d^3p\, d^3q,$$

$$\mathcal{E}_q(f) \equiv -\frac{1}{2} \iiiint \frac{f(p',q')f(p,q)}{|q-q'|}\, d^3p'\, d^3p\, d^3q'\, d^3q.$$

Main Steps in the Determination of $\mathcal{C}_{\beta}^{-}$

1. Show the existence of minimizers. This is accomplished by suitably altering an argument of M. Weinstein.
2. Characterize the minimizers. The minimizers are found to be given by the famous Lane-Emden functions.
3. Compute $\mathcal{C}_{\beta}^{-}$ (which will depend on the first zeroes and derivatives at the first zero of the Lane-Emden functions). This last step must be done numerically as very few of the Lane-Emden functions are given by nice expressions.

Existence of Minimizers

1. It is easier to minimize over an expanded class of functions:

$$\Omega_\beta = \{f : \mathbb{R}^6 \to \mathbb{R} : f \geq 0, \|f\|_1 + \||p|f\|_1 + \|f\|_\beta < \infty\}.$$

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2. We have to make a corresponding change in our functional

$$\widetilde{\Phi}_\beta(f) \equiv \left(\frac{\mathcal{E}_p^u(f)}{-\mathcal{E}_q(f)}\right)^{3(1-\frac{1}{\beta})} \|f\|_\beta \, \|f\|_1^{2-\frac{3}{\beta}},$$

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   with infimum $\widetilde{\mathcal{C}}_\beta$ over Ω_β.
3. The triple family of scalings

$$f_{\kappa,\lambda,\mu}(p,q) \equiv \mu\, f(\lambda p, \kappa q)$$

   leaves $\widetilde{\Phi}_\beta$ invariant and allows us to choose a minimizing sequence $f_{\beta,n}$ so that $\widetilde{\Phi}_\beta\,(f_{\beta,n}) = (-\mathcal{E}_q(f_{\beta,n}))^{-3(1-\frac{1}{\beta})}$. By taking spherically symmetric equi-measurable rearrangements, we can also assume our functions are spherically symmetric.

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$$\lim_{n\to\infty} \left(-\mathcal{E}_q(f_{\beta,n})\right)^{-3(1-\frac{1}{\beta})} = \widetilde{\mathcal{C}}_{\beta} \leq \left(-\mathcal{E}_q(f_{\beta})\right)^{-3(1-\frac{1}{\beta})}.$$

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6. Several technical estimates show

$$\lim_{n\to\infty} \mathcal{E}_q(f_{\beta,n}) = \mathcal{E}_q(f_\beta)$$

giving that f_β is a minimizer for $\widetilde{\Phi}_\beta$ over Ω_β.

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$$\frac{d}{dt}\bigg|_{t=0^+} \widetilde{\Phi}_\beta((1-t)f_\beta + t\eta).$$

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This has an advantage over using arbitrary variations of f_β, which may not belong to Ω_β.
2. After computing derivatives and taking advantage of the scaling above, we get
$$f_\beta(p,q) = (\phi_\beta(q) - |p|)_+^{\frac{1}{\beta-1}},$$
where $\phi_\beta(q)$ satisfies the Lane-Emden PDE:
$$-\triangle\phi_\beta(q) = c(\beta)\,(\phi_\beta(q))_+^{\frac{3\beta-2}{\beta-1}}$$
for a constant $c(\beta)$.

Characterization of Minimizers

3. By spherical symmetry, this becomes the following ODE

$$\begin{aligned} \frac{d^2\phi_\beta}{dr^2} + \frac{2}{r}\frac{d\phi_\beta}{dr} + c(\beta)\,(\phi_\beta)_+^{\frac{3\beta-2}{\beta-1}} &= 0, \\ \frac{d\phi_\beta}{dr}(0) &= 0, \\ \frac{d\phi_\beta}{dr}(R_\beta) &= -\frac{1}{R_\beta^2}. \end{aligned}$$

The first boundary condition is imposed by the requirement that f_β be finite at the origin. The second one ensures that f_β has total mass 1 and is $\mathfrak{C}^1$. Here, R_β is the first zero of $\phi_\beta(r)$.

Characterization of Minimizers

4. When $\beta > 3/2$, the exponent appearing in the ODE, n_β, satisfies

$$3 < n_\beta = \frac{3\beta - 2}{\beta - 1} < 5,$$

with $\beta = 3/2$ giving $n_\beta = 5$ and $\beta \to \infty$ limiting to $n_\beta = 3$.
As is well known, for $0 \leq n_\beta < 5$, the solutions to the Lane-Emden ODE have a zero at a finite distance from the origin. The $\beta = 3/2$ (and so, $n_\beta = 5$) case does not have compact support (this solution is known as Plummer's Sphere).

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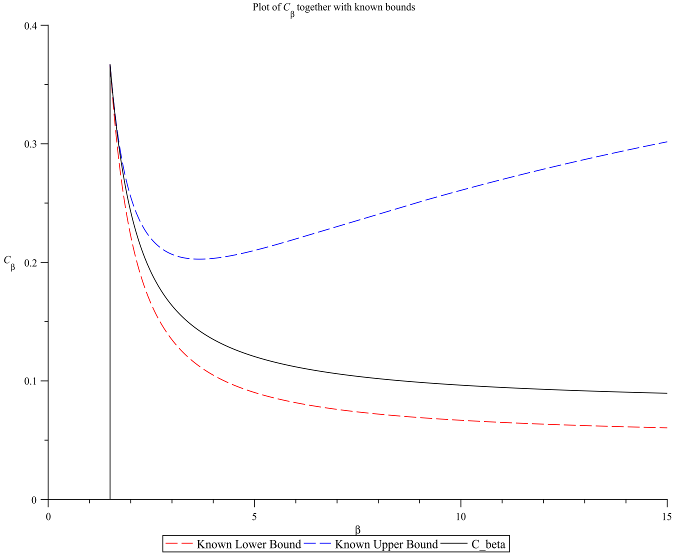
5. Except in the critical case $\beta = 3/2$ (which was found explicitly in the paper of Kiessling and Tahvildar-Zadeh), we find that our minimizers for $\widetilde{\mathcal{C}}_\beta$ are actually in $\mathfrak{P}_1 \cap \mathfrak{L}^\beta$ and so give us $\mathcal{C}_\beta^-$.

$$\mathcal{C}_\beta^- = \left(\frac{\beta}{R_\beta(2\beta - 3)} \right)^{\frac{1}{\beta}}.$$

Numerical Results



Asymptotics

Using Buchdahl's result (1978) on the Plummer's Sphere ($\beta = 3/2$) we find an asymptotic formula for $\mathcal{C}_{\beta}^{-}$ that works well for $\beta \searrow 3/2$:

$$\mathcal{C}_{\beta}^{-} \approx \left[\frac{3\pi}{16}\left(\frac{\beta}{4\beta-3}\right)\right]^{\frac{1}{\beta}}\left(\frac{3\beta(2\beta-1)(3\beta-2)}{32\pi^{2}(\beta-1)^{3}}\right)^{1-\frac{1}{\beta}}.$$

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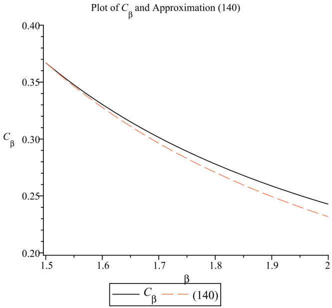

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- In 2008, Kiessling and Tahvildar-Zadeh proved that any spherically symmetric classical solution of rVP$^-$ launched by initial data $f_0 \in \mathfrak{P}_3 \cap \mathfrak{C}^1$ with *zero total energy* and total virial less than or equal to -1/2 will blow up in finite time.

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- S. Calogero (private communication to S. Tahvildar-Zadeh) has questioned the existence of such initial data.
- We show the existence of zero-energy initial data that lead to finite-time blow-up for rVP$^-$.

Nearly Uniform Balls have Virial $> -1/2$

- Throughout, we use functions which are not $\mathfrak{C}^1$ to make calculations easier. We do this with the understanding that an appropriate procedure can be employed to regularize the data at the end of the construction.

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- Throughout, we use functions which are not $\mathfrak{C}^1$ to make calculations easier. We do this with the understanding that an appropriate procedure can be employed to regularize the data at the end of the construction.
- First, the simplest possible ansatz will not work! That is, take

$$f(p,q) = \mathcal{C}\eta(|q|)\Phi(|p|)\mathscr{L}(\cos(\theta_{p,q})),$$

with

$$\begin{aligned} \eta(|q|) &= \chi_{[0,R]}(|q|), \\ \Phi(|p|) &= \chi_{[0,P]}(|p|), \\ \mathscr{L}(x) &= \chi_{[-1,a]}(x) \end{aligned}$$

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- The zero energy requirement forces us to equate

$$\begin{aligned}\mathcal{E}_p(f) &= \frac{3}{8}\left(\frac{\sqrt{1+P^2}}{P^2} + 2\sqrt{1+P^2} - \frac{\ln(P+\sqrt{1+P^2})}{P^3}\right), \\ \mathcal{E}_q(f) &= -\frac{3}{5R},\end{aligned}$$

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  giving us $R = R(P)$.
- This allows us to compute the virial $\mathcal{V}(f)$ and we find for any choice of P and a

$$\mathcal{V}(f) > -\frac{9}{20} > -\frac{1}{2}.$$

A Core-Halo Ansatz

Next, we try $f(p,q) = \mathcal{C}\eta(|q|)\Phi(|p|)\mathscr{L}(\cos(\theta_{p,q}))$, with

$$\begin{aligned}
\eta(|q|) &= \chi_{[0,R_1]}(|q|) + \alpha\chi_{[R_2,R_3]}(|q|), \\
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- Forcing the energy to be zero gives a complicated (but positive) expression for α.
- Choosing $-1 < a \leq -4/5$ gives a virial which is less than $-1/2$.

A Core-Halo Ansatz

- We can even choose our parameters in such a way that the virial becomes arbitrarily negative.

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and the zero energy condition allows us to solve for $\alpha = \alpha(P)$. We find that α will be positive for sufficiently large P and is asymptotically proportional to $P^{-23/2}$.

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  and the zero energy condition allows us to solve for $\alpha = \alpha(P)$. We find that α will be positive for sufficiently large P and is asymptotically proportional to $P^{-23/2}$.
- Plugging these choices into the formula for the virial and looking at the asymptotics for large P shows that the virial is proportional to $-(1-a)P^3$.

A Monotonically Decreasing Core-Halo Ansatz

We now report on a second class of favorable initial data.
Again, take $f(p,q) = \mathcal{C}\eta(|q|)\Phi(|p|)\mathscr{L}(\cos(\theta_{p,q}))$, but with

$$\begin{aligned} \eta(|q|) &= \chi_{[0,R_1]}(|q|) + \left(\frac{R_1}{|q|}\right)^n \chi_{[R_1,R_2]}(|q|) + \left(\frac{R_1}{R_2}\right)^n \chi_{[R_2,R_3]}(|q|), \\ \Phi(|p|) &= \chi_{[0,P]}(|p|), \\ \mathscr{L}(x) &= \chi_{[-1,a]}(x). \end{aligned}$$

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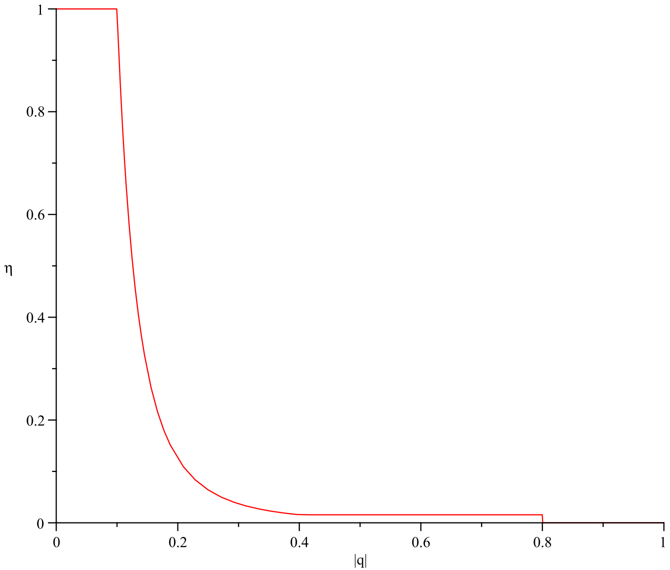
- It is also monotonically decreasing.
- Choosing
$$R_1 = \frac{1}{100}, R_2 = \frac{1}{11}, R_3 = \frac{1}{10}, n = 3$$
sets a corresponding P (about 19.69) to force the zero-energy condition. Any choice of a less than roughly $-9/10$ gives us a virial less than $-1/2$.

A Monotonically Decreasing Core-Halo Ansatz



Outline for Remainder of Talk

- A "Relativistic" 2N-Body Coulomb System
- A Regularized Version of rVP$^-$
- Large Deviations, Entropy, and
  Rates of Convergence for Initial Data
- Space-Time Rescalings
- The Traditional Vlasov Space-Time Scale
- The *A Priori* Space-Time Scale
- Conclusions and Future Directions

A "Relativistic" 2N-Body Coulomb System

- Fix a *spherically symmetric* regularizer $\eta \in \mathscr{C}_c^\infty(\mathbb{R}^3)$ with

$$0 \le \eta \le 1, \|\eta\|_1 = 1, \text{ and } \operatorname{supp}(\eta) \subseteq B_1(0).$$

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$$G_\epsilon(q_1, q_2) \;\; = \;\; \iint \eta_\epsilon(q_1 - w)\frac{w - w'}{|w - w'|^3}\eta_\epsilon(w' - q_2)d^3wd^3w'.$$

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We consider an overall neutral two-specie charged plasma with N particles of each type. All particles have unit mass. Even labels will refer to the positively charged species; odd labels will refer to the negative charges.

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- The dynamics is given by:

$$\begin{aligned} \dot{q}_i(t) &= v(p_i(t)), \\ \dot{p}_i(t) &= e_i \sum_{j=1}^{N} G_\epsilon(q_i(t), q_{2j}(t)) - G_\epsilon(q_i(t), q_{2j-1}(t)), \end{aligned}$$

where $v(p) = \frac{p}{\sqrt{1+|p|^2}}$.

A “Relativistic” $2N$-Body Coulomb System

We assume the initial condition

$$\mathcal{X}(0) \equiv (q_1(0), p_1(0), \dots, q_{2N}(0), p_{2N}(0)) \in \mathbb{R}^{12N}$$

for our plasma is chosen randomly according to

$$\mathbb{P}_0 = \bigotimes_{i=1}^{2N} f_0 \, d^3p \, d^3q,$$

where $f_0 \geq 0$ is any sufficiently nice function (continuously differentiable, say) on $\mathbb{R}^6$ with

$$\iint f_0(p, q) \, d^3p \, d^3q = 1.$$

A "Relativistic" 2N-Body Coulomb System

We define the following *empirical measures*:

$$\begin{aligned}
{}^{N}\Delta_t^{+}(p,q) &= \frac{1}{N}\sum_{i=1}^{N}\delta(p-p_{2i}(t))\delta(q-q_{2i}(t)),\\
{}^{N}\rho_t^{+}(q) &= \frac{1}{N}\sum_{i=1}^{N}\delta(q-q_{2i}(t)),\\
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$$

Using the densities, our dynamics is given by a coupled pair of PDEs:

$$\partial_t {}^N\Delta_t^\pm + v(p)\cdot\nabla_q {}^N\Delta_t^\pm \pm NG_\epsilon * \left[{}^N\rho_t^+ - {}^N\rho_t^-\right](q)\cdot\nabla_p {}^N\Delta_t^\pm = 0.$$

A Regularized Version of rVP$^-$

- Let f_t satisfy

$$\partial_t f_t + v(p) \cdot \nabla_q f_t - G_\epsilon * \rho_{f_t}(q) \cdot \nabla_p f_t = 0,$$

where

$$\rho_{f_t}(q) = \int f_t(p, q) d^3 q,$$

launched by some sufficiently regular f_0.

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  launched by some sufficiently regular f_0.
- Associated to this PDE is a flow on $\mathbb{R}^6$

$$\begin{aligned} V^0[f_\cdot](t,p,q) &= \begin{bmatrix} -G_\epsilon * \rho_{f_t}(q) \\ v(p) \end{bmatrix} \\ T^0_{t,0}[f_\cdot](p,q) &= (p(t),q(t)) \\ \begin{bmatrix} \dot{p}(t) \\ \dot{q}(t) \end{bmatrix} &= V^0[f_\cdot](t,p(t),q(t)) \\ (p(0),q(0)) &= (p,q). \end{aligned}$$

A Regularized Version of rVP$^-$

We list some basic properties of our flow.

- First, a sharp version of the HLS Inequality shows us that G_ϵ is bounded and Lipschitz continuous (in either slot) with common bound

$$L_G = \frac{1}{\epsilon^3}\left(\frac{8\pi^2}{3}\right)^{\frac{4}{3}} L_\eta.$$

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- This gives us a t-independent Lipschitz constant for $V^0[f.]$

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- By a standard application of Gronwall's Inequality
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Large Deviations, Entropy, and Rates of Convergence for Initial Data

The topology we use is given by the dual, bounded Lipschitz distance (denoted d_{bL^*}) on the space of probability measures: for two probability measures μ and ν defined on $\mathbb{R}^6$

$$d_{bL^*}(\mu,\nu) = \sup_{\varphi \in \mathcal{D}(\mathbb{R}^6)} \left| \int \varphi d(\mu - \nu) \right|,$$

where

$$\mathcal{D}(\mathbb{R}^6) = \left\{ \varphi : \mathbb{R}^6 \to [-1,1] \mid \mathrm{Lip}(\varphi) \leq 1 \right\}.$$

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This topology is equivalent to the topology generated by all bounded, continuous functions. Probabilists call convergence w.r.t. d_{bL^*} *convergence in law*.

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- $H(\mu|f_0 d^3p d^3q)$ is the relative entropy of μ w.r.t. $f_0 d^3p d^3q$:

$$H(\mu|\nu) \equiv \int \ln\left(\frac{d\mu}{d\nu}\right) d\mu = \int \frac{d\mu}{d\nu} \ln\left(\frac{d\mu}{d\nu}\right) d\nu$$

Large Deviations, Entropy, and Rates of Convergence for Initial Data

Since $d_{bL^*}({}^N\Delta_0^{\pm}, f_0)$ is a continuous function of the random variable ${}^N\Delta_0^{\pm}$, the *Contraction Principle* states that the metric also satisfies a large deviation principle:

$$\mathbb{P}_0\left(d_{bL^*}({}^N\Delta_0^{\pm}, f_0) > \delta\right) \asymp \exp\left(-\underline{\mathcal{H}}_{f_0}(\delta)N\right)$$

with rate function

$$\underline{\mathcal{H}}_{f_0}(\delta) \quad \equiv \quad \inf_{\mu\in M_1(\mathbb{R}^6)}\{H(\mu|f_0 d^3p d^3q) \ : \ d_{bL^*}(\mu, f_0) > \delta\}$$

for any $\delta > 0$.

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- We assume $f_0 d^3p d^3q$ satisfies the inequality LSI(κ, λ):

$$H(\nu | f_0 d^3p d^3q) \leq \frac{1}{(1+1/\kappa)\lambda^{1/\kappa}} \int \left| \nabla \ln \frac{d\nu}{df_0} \right|^{1+1/\kappa} d\nu$$

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- A lengthy argument shows $f_0 d^3p d^3q$ satisfies LSI(κ, λ) iff

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- We emphasize that κ must satisfy $\kappa \geq 1$.

Large Deviations, Entropy, and Rates of Convergence for Initial Data

- Suppose that $f_0 = \exp(-V)$ for some convex function V. If V satisfies the condition

$$V(x) + V(y) - 2V\left(\frac{x+y}{2}\right) \geq \frac{\lambda}{\kappa}\|x-y\|^{1+\kappa},$$

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- For $\kappa = 1$, any V of the form

$$V(p,q) = g(p,q) + |(p,q)|^2$$

  with g convex will satisfy LSI$(1, \lambda)$ for some λ.

Large Deviations, Entropy, and Rates of Convergence for Initial Data

Assuming f_0 satisfies LSI(κ, λ) gives the following rate

$$\mathbb{P}_0 \left(d_{bL^*}({}^N\Delta_0^{\pm}, f_0) > \delta \right) \quad \lesssim \quad \exp\left(-\frac{\lambda \delta^{1+\kappa}}{1+\kappa} N \right)$$

which shows that at $t = 0$, $d_{bL^*}({}^N\Delta_0^{\pm}, f_0)$ converges to zero in probability.

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For sufficiently large N, the expected distance converges to zero like:

$$\mathbb{E}_{\mathbb{P}_0} \left[d_{bL^*}({}^N\Delta_0^{\pm}, f_0) \right] \quad \lesssim \quad \left(\frac{1}{\lambda N} \right)^{\frac{2}{1+\kappa}} \Gamma\left(\frac{2}{1+\kappa} \right) (1+\kappa)^{\frac{1-\kappa}{1+\kappa}}.$$

Space-Time Rescalings

Recall that our empirical-measure-dynamics is given by

$$\partial_t {}^N\!\Delta_t^{\pm} + v(p) \cdot \nabla_q {}^N\!\Delta_t^{\pm} \pm N G_\epsilon * \left[{}^N\!\rho_t^{+} - {}^N\!\rho_t^{-} \right](q) \cdot \nabla_p {}^N\!\Delta_t^{\pm} = 0.$$

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- Many treatments assume the mass of the constituent particles is $1/N$. This will not work well with the relativistic velocity.
- We choose to rescale space and time variables with N:

$$\begin{aligned} \bar{t} &= N^\alpha t, \\ \bar{q} &= N^\alpha q, \\ \bar{\epsilon} &= N^\alpha \epsilon, \\ \bar{p} &= p. \end{aligned}$$

Space-Time Rescalings

- The rescaled normalized densities are given by

$$ {}^{N}\bar{\Delta}^{\pm}_{\bar{t}}(\bar{p},\bar{q}) = N^{-3\alpha}\ \ {}^{N}\Delta^{\pm}_{N^{-\alpha}\bar{t}}\left(\bar{p}, N^{-\alpha}\bar{q}\right) . $$

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$$ \partial_{\bar{t}}{}^{N}\bar{\Delta}^{\pm}_{\bar{t}} + v(\bar{p}) \cdot \nabla_{\bar{q}}{}^{N}\bar{\Delta}^{\pm}_{\bar{t}} \pm N^{1+\alpha} G_{\bar{\epsilon}} * \left[{}^{N}\bar{\rho}^{+}_{t} - {}^{N}\bar{\rho}^{-}_{t}\right](\bar{q}) \cdot \nabla_{\bar{p}}{}^{N}\bar{\Delta}^{\pm}_{\bar{t}} = 0. $$

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$$ \partial_{\bar{t}}{}^N\bar{\Delta}^{\pm}_{\bar{t}} + v(\bar{p})\cdot\nabla_{\bar{q}}{}^N\bar{\Delta}^{\pm}_{\bar{t}} \pm N^{1+\alpha}G_{\bar{\epsilon}} * \left[{}^N\bar{\rho}^{+}_{t} - {}^N\bar{\rho}^{-}_{t}\right](\bar{q})\cdot\nabla_{\bar{p}}{}^N\bar{\Delta}^{\pm}_{\bar{t}} = 0. $$

- The choice $\alpha = -1$ corresponds to the *traditional Vlasov scales* (here, long time and large space scales).

Space-Time Rescalings

- The rescaled normalized densities are given by

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- The choice $\alpha = -1$ corresponds to the *traditional Vlasov scales* (here, long time and large space scales).
- The choice $\alpha = 0$ gives the *a priori* scales (where Vlasov limits are not traditionally studied).

The Traditional Vlasov Space-Time Scale

To be definite, our system of PDEs on the $\alpha = -1$ scale is

$$\partial_t {}^N\!\Delta_t^{\pm} + v(p) \cdot \nabla_q {}^N\!\Delta_t^{\pm} \pm G_\epsilon * \left[{}^N\!\rho_t^{+} - {}^N\!\rho_t^{-} \right](q) \cdot \nabla_p {}^N\!\Delta_t^{\pm} = 0.$$

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- We show ${}^N\Delta_t^{\pm} \to f_t^{\pm}$ as $N \to \infty$, where $f_t^{\pm}$ satisfy the following *two-specie relativistic Vlasov-Poisson system*:

$$\partial_t f_t^{\pm} + v(p) \cdot \nabla_q f_t^{\pm} \pm G_\epsilon * \left[\rho_{f_t^+} - \rho_{f_t^-} \right] (q) \cdot \nabla_p f_t^{\pm} = 0,$$

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- In the special case $f_0^+ = f_0^- = f_0$, then our coupled PDEs reduce to the free-streaming PDE:

$$\partial_t f_t + v(p) \cdot \nabla_q f_t = 0$$

which has the obvious solution $f_t(p,q) = f_0(p, q - tv(p))$.

The Traditional Vlasov Space-Time Scale

The main estimate showing convergence for all t is

$$d_{bL^*}({}^N\Delta_t^+, f_t^+) + d_{bL^*}({}^N\Delta_t^-, f_t^-) \leq \frac{e^{(2\sqrt{3}L_G + L_\pm)t}}{2\sqrt{3}L_G} \left(d_{bL^*}\left({}^N\Delta_0^+, f_0^+\right) + d_{bL^*}\left({}^N\Delta_0^-, f_0^-\right) \right).$$

where $L_\pm = \max\{1, 2L_G\}$.

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- Sanov's Theorem still applies, and so

$$\mathbb{E}_{\mathbb{P}_0}\left[d_{bL^*}({}^N\Delta_t^+, f_t^+) + d_{bL^*}({}^N\Delta_t^-, f_t^-)\right] \lesssim \frac{\mathcal{C}e^{(2\sqrt{3}L_G + L_\pm)t}}{L_G N^{\frac{2}{1+\kappa}}}.$$

The *A Priori* Space-Time Scale

For concreteness, our system of PDEs on the $\alpha = 0$ scale is

$$\partial_t {}^N\!\Delta_t^{\pm} + v(p) \cdot \nabla_q {}^N\!\Delta_t^{\pm} \pm N G_\epsilon * \left[{}^N\!\rho_t^{+} - {}^N\!\rho_t^{-} \right](q) \cdot \nabla_p {}^N\!\Delta_t^{\pm} = 0.$$

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- We now consider initial data chosen *iid* by f_0 jointly in both species (recall, this gives free-streaming on the $\alpha = -1$ scale).
- Associated to this PDE system are flows on $\mathbb{R}^6$:

$$\begin{aligned}
V^\pm \left[{}^N\!\Delta_\cdot^+, {}^N\!\Delta_\cdot^- \right](t,p,q) &= \begin{bmatrix} \pm N G_\epsilon * \left[{}^N\!\rho_t^+ - {}^N\!\rho_t^- \right](q) \\ v(p) \end{bmatrix}, \\
T_{t,0}^\pm \left[{}^N\!\Delta_\cdot^+, {}^N\!\Delta_\cdot^- \right](p,q) &= (p(t), q(t)), \\
\begin{bmatrix} \dot{p}(t) \\ \dot{q}(t) \end{bmatrix} &= V^\pm \left[{}^N\!\Delta_\cdot^+, {}^N\!\Delta_\cdot^- \right](t, p(t), q(t)), \\
(p(0), q(0)) &= (p,q).
\end{aligned}$$

The *A Priori* Space-Time Scale

- Recall, we want to compare the finite N dynamics to f_t given by the regularized version of rVP$^-$:

$$\partial_t f_t + v(p) \cdot \nabla_q f_t - G_\epsilon * \rho_{f_t}(q) \cdot \nabla_p f_t = 0.$$

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- Recall that this PDE generates a flow $T^0_{t,0}[f_\cdot]$ with associated vector field $V^0[f_\cdot]$, which has Lipschitz constant L_0 (depending on L_G), and so $T^0_{t,0}[f_\cdot]$ has Lipschitz constant $e^{L_0 t}$.
- $L_0 = \max\{1, L_G\}$.

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Using the various flows we have defined, we can represent our PDEs as fixed point equations.

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- For the regularized version of rVP$^-$:

$$ f_t(p,q) \quad = \quad f_0 \circ T_{0,t}^{0}[f_{\cdot}](p,q). $$

Basic Calculations

We use the definition of d_{bL^*} and the basic properties of test functions $\varphi \in \mathcal{D}(\mathbb{R}^6)$ to find:

$$d_{bL^*}({}^N\Delta_t^+, f_t) + d_{bL^*}({}^N\Delta_t^-, f_t) \leq e^{L_0 t}\left(d_{bL^*}({}^N\Delta_0^+, f_0) + d_{bL^*}({}^N\Delta_0^-, f_0)\right) + \lambda^+(t) + \lambda^-(t).$$

where

$$\lambda^{\pm}(t) = \iint \left| T_{t,0}^{\pm}\left[{}^N\Delta_\cdot^+, {}^N\Delta_\cdot^-\right](p,q) - T_{t,0}^{0}[f_\cdot](p,q)\right| {}^N\Delta_0^{\pm}(p,q)d^3pd^3q.$$

Basic Calculations

Iterating the flow and using Gronwall's Inequality gives

$$\lambda^{\pm}(t) \leq e^{L_0 t} \int_0^t \gamma^{\pm}(\tau) e^{-L_0 \tau} d\tau$$

where

$$\begin{aligned}\gamma^{\pm}(t) &= \iint \left| V^{\pm}\left[{}^N\Delta_{\cdot}^{+}, {}^N\Delta_{\cdot}^{-}\right](t,p,q) - V^0[f_{\cdot}](t,p,q) \right| {}^N\Delta_t^{\pm}(p,q) d^3p d^3q \\ &= \iint \left| G_\epsilon * \left[\pm N {}^N\rho_t^{+} \mp N {}^N\rho_t^{-} + \rho_{f_t} \right](q) \right| {}^N\Delta_t^{\pm}(p,q) d^3p d^3q.\end{aligned}$$

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If we focus on $\gamma^{+}(t)$, we see

$$\gamma^{+}(t) = \sum_{i=1}^{N} \left| G_\epsilon * \left[{}^N\rho^{+}_t - {}^N\rho^{-}_t + \frac{1}{N}\rho_{f_t}\right](q_{2i}(t))\right|.$$

Basic Calculations

We average over removing the j-th negative particle:

$$\begin{aligned}
\gamma^+(t) &= \sum_{i=1}^{N} \left| G_\epsilon * \left[\frac{N-1}{N} {}^{N\setminus i}\rho_t^+ - {}^{N}\rho_t^- + \frac{1}{N}\rho_{f_t} \right] (q_{2i}(t)) \right| \\
&\leq \frac{N-1}{N^2} \sum_{i=1}^{N} \sum_{j=1}^{N} \left| G_\epsilon * \left[{}^{N\setminus i}\rho_t^+ - {}^{N\setminus j}\rho_t^- \right] (q_{2i}(t)) \right| \\
&\quad + \frac{1}{N} \sum_{i=1}^{N} \left| G_\epsilon * \left[\rho_{f_t} - {}^{N}\rho_t^- \right] (q_{2i}(t)) \right|
\end{aligned}$$

where

$$\begin{aligned}
{}^{N\setminus j}\Delta_t^+(p,q) &= \frac{1}{N-1} \sum_{k\neq j} \delta(p - p_{2k}(t))\delta(q - q_{2k}(t)), \\
{}^{N\setminus j}\Delta_t^-(p,q) &= \frac{1}{N-1} \sum_{k\neq j} \delta(p - p_{2k-1}(t))\delta(q - q_{2k-1}(t)).
\end{aligned}$$

Basic Calculations

Using the fact that G_ϵ is a vector-valued function with components that are bounded and Lipschitz continuous (by L_G) gives:

$$\gamma^+(t) \leq \sqrt{3} L_G \frac{N-1}{N} \sum_{i=1}^{N} \left[d_{bL^*}\left({}^{N\setminus i}\Delta_t^+, f_t \right) + d_{bL^*}\left({}^{N\setminus i}\Delta_t^-, f_t \right) \right]$$
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Ultimately, we will only look at *expectation values* over our ensemble. Since all particle labels are arbitrary in the ensemble, we have

$$\mathbb{E}_{\mathbb{P}_0}\left[\gamma^+(t)\right] \leq \sqrt{3} L_G (N-1) \mathbb{E}_{\mathbb{P}_0}\left[d_{bL^*}\left({}^{N-1}\Delta_t^+, f_t \right) + d_{bL^*}\left({}^{N-1}\Delta_t^-, f_t \right) \right] + \sqrt{3} L_G \mathbb{E}_{\mathbb{P}_0}\left[d_{bL^*}\left({}^{N}\Delta_t^-, f_t \right) \right].$$

Basic Calculations

Putting this all together with previous estimates (and realizing that in the limit $N-1$ particles at time t is not essentially different than having N particles) gives:

$$
\begin{aligned}
e^{-(\sqrt{3}L_G+L_0)t}\mathbb{E}_{\mathbb{P}_0}\left[d_{bL^*}({}^N\Delta_t^+,f_t)+d_{bL^*}({}^N\Delta_t^-,f_t)\right]
&\lesssim \mathbb{E}_{\mathbb{P}_0}\left[d_{bL^*}({}^N\Delta_0^+,f_0)+d_{bL^*}({}^N\Delta_0^-,f_0)\right] \\
&\quad + 2\sqrt{3}L_G N\int_0^t \mathbb{E}_{\mathbb{P}_0}\left[d_{bL^*}\left({}^N\Delta_\tau^+,f_\tau\right)+d_{bL^*}\left({}^N\Delta_\tau^-,f_\tau\right)\right] \\
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A final application of Gronwall's Inequality gives:

$$
\begin{aligned}
\mathbb{E}_{\mathbb{P}_0}\left[d_{bL^*}({}^N\Delta_t^+, f_t) + d_{bL^*}({}^N\Delta_t^-, f_t)\right] \\
\lesssim e^{2\sqrt{3}L_0 N t}\mathbb{E}_{\mathbb{P}_0}\left[d_{bL^*}({}^N\Delta_0^+, f_0) + d_{bL^*}({}^N\Delta_0^-, f_0)\right]
\end{aligned}
$$

Final Large Deviation Results

Combining the previous inequality with our expected rates of convergence from Sanov's Theorem gives:

$$\mathbb{E}_{\mathbb{P}_0}\left[d_{bL^*}({}^N\Delta_t^+, f_t) + d_{bL^*}({}^N\Delta_t^-, f_t)\right] \lesssim \frac{Ce^{2\sqrt{3}L_0Nt}}{N^{\frac{2}{1+\kappa}}}.$$

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- While we can't take the limit $N \to \infty$ for any $t > 0$, we can conclude that for *finite* (but large) N our regularized version of rVP$^-$ approximates our dynamics reasonably well for times on the order of $1/N$.
- We could get convergence at all later times by requiring $L_G N$ to tend to some finite quantity as N tends to infinity. This would force $N\epsilon^{-3}$ to limit to something finite - or ϵ to grow on the order of $N^{1/3}$. This is an extremely unphysical state of affairs.

Question: What is going on?

I. One possibility is that rVP$^-$ is giving the dynamics for f_t, but our estimates are not sharp enough to show it.
II. We outline what seems to be a more likely possibility. Consider the expected value of the force (as felt by any one of the negative particles, say).

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- At the initial time, the expected force term is *exactly* the force from our regularized version of rVP$^-$:

$$\mathbb{E}_{\mathbb{P}_0}\left[-G_\epsilon * \left[N^N\rho_0^+ - (N-1)^{N-1}\rho_0^-\right](q)\right] = -G_\epsilon * \rho_{f_0}(q).$$

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- Provided that empirical one-point densities at time t converge to a deterministic f_t, then the rVP$^-$ force term results again:

$$\mathbb{E}_{\mathbb{P}_t}\left[-G_\epsilon * \left[N^N\rho_t^+ - (N-1)^{N-1}\rho_t^-\right](q)\right] \quad = \quad -G_\epsilon * \rho_{f_t}(q).$$

However, this *does not* imply that f_t satisfies rVP$^-$.

Question: What is going on?

To shed more light on the matter, we consider the discrepancy of the force from its expected value:

$$\left| G_\epsilon * \left[-\rho_{f_0} + N{}^N\!\rho_0^+ - (N-1){}^{N-1}\!\rho_0^- \right](q) \right| .$$

For f_0 satisfying LSI(κ, λ), Large Deviation techniques give:

$$\mathbb{E}_{\mathbb{P}_0}\left[\left| G_\epsilon * \left[-\rho_{f_0} + N{}^N\!\rho_0^+ - (N-1){}^{N-1}\!\rho_0^- \right](q) \right| \right] \asymp \frac{\mathcal{C}}{N^{\frac{1-\kappa}{1+\kappa}}}$$

$$\mathrm{Var}_{\mathbb{P}_0}\left(\left| G_\epsilon * \left[-\rho_{f_0} + N{}^N\!\rho_0^+ - (N-1){}^{N-1}\!\rho_0^- \right](q) \right| \right) \asymp \frac{\mathcal{C}}{N^{\frac{2-2\kappa}{1+\kappa}}}.$$

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Since for $\kappa = 1$ these quantities are bounded as $N \to \infty$, the force discrepancy term may actually converge in distribution in this case.

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3. Another possibility based on the analysis of the discrepancy is that there is some stochastic PDE, containing the rVP$^-$ force term, that controls the dynamics of f_t.

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3. Another possibility based on the analysis of the discrepancy is that there is some stochastic PDE, containing the rVP^- force term, that controls the dynamics of f_t.

NB: In the case of a 2. or 3., time averages over some window of size T for the spatial densities may well be approximated by those from rVP^-.

Conjecture

We conjecture that the dynamics for f_t is given by

$$\partial_t f_t + v(p) \cdot \nabla_q f_t - G_\epsilon * \rho_{f_t}(q) \cdot \nabla_p f_t = \mathfrak{C}(f_t),$$

where $\mathfrak{C}$ represents whatever “collision” operator is influencing the dynamics.

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Assuming that $\mathfrak{C}(f) = 0$ if and only if f is the Boltzmann-Jüttner distribution

$$f_J(p,q) = \mathcal{C} e^{-\beta\left(\sqrt{1+|p|^2}+\psi(q)\right)},$$

(which is the case for all the known operators we have in mind), then any stationary state of our PDE is given by f_J and will satisfy the *stationary* relativistic Vlasov-Poisson system:

$$v(p) \cdot \nabla_q f_J - G_\epsilon * \rho_{f_J}(q) \cdot \nabla_p f_J = 0.$$

We plan to explore this conjecture in future work.