STUDIES OF THE RELATIVISTIC VLASOV-POISSON SYSTEM

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ABSTRACT OF THE DISSERTATION

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We examine several questions pertaining to the relativistic Vlasov-Poisson system with attractive coupling (rVP$^-$) raised in a recent paper [KTZ08] by Kiessling and Tahvildar-Zadeh (KTZ). First, KTZ proved that for every $\beta \geq 3/2$ there is a critical, non-zero value $\mathcal{C}_\beta^-$ such that certain initial data with $\mathfrak{L}^\beta$ norm less than $\mathcal{C}_\beta^-$ launch solutions to rVP$^-$ which exist globally in time. The authors obtained the sharp value for $\mathcal{C}_{3/2}^-$ and characterized the remaining constants via a minimization problem. We show the existence of minimizers and calculate $\mathcal{C}_\beta^-$ for $\beta > 3/2$.

Second, KTZ proved that any spherically symmetric classical solution of rVP$^-$ launched by zero energy initial data with virial $\leq -1/2$ will blow up in finite time. However, Simone Calogero has raised the question whether any such data exist at all. We settle this question by constructing two different classes of such initial data.

Third, we examine the recent proposal in [KTZ08] whereby rVP$^-$ might be derived from an overall neutral two-specie, spherically symmetric plasma with initial condition chosen *iid* in both species, interacting through regularized electromagnetic fields *on space-time scales not typically considered in Vlasov-type limits.* We show first that on the usual scales the familiar relativistic Vlasov-Poisson system for an overall neutral two-specie plasma is obtained if the particles of each specie are separately chosen *iid* by f_0^+ and f_0^- respectively. If $f_0^+ = f_0^-$ this dynamics reduces to trivial free-streaming of

all particles, with $f_t^+ = f_t^-$ for all later times (on this Vlasov scale). To see non-trivial plasma dynamics, the usual procedure would be to look at longer time scales and to correct the dynamics by adding a "relativistic" generalization of the Lenard-Balescu "collision" operator to the free-streaming Vlasov operator. The proposal in [KTZ08] is that instead of the collision operator, the rVP$^-$ force term of a single specie Newton system could emerge on the *a priori* scales. We examine this proposal for the simpler case of purely Coulombic interactions. We will extract from this reduced model evidence for *both* an rVP$^-$ force term and a dissipative operator that govern the dynamics.

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First and foremost, many thanks go to Michael Kiessling for so much more than I could possible relate here. I could not have asked for a more knowledgeable and patient advisor. Great thanks also go to my committee members Eric Carlen, Herbert Spohn, and Shadi Tahvildar-Zadeh for their support and comments on this manuscript. I was also fortunate to receive funding through NSF grant DMS 08-07705 awarded to Michael Kiessling.

The results in Chapter 2 were first presented as a poster at the conference Vlasovia III held in Marseilles, France in 2009 and will appear in the journal *Transport Theory and Statistical Physics* in 2011. Thanks go to Yves Elskens and the other organizers of Vlasovia III for a stimulating meeting, and special thanks to Vlasovia III for providing supplementary financial support. Thanks also go to two anonymous referees for their very helpful comments on this material.

Finally, the results of Chapter 3 answer the question of Simone Calogero of whether zero energy initial data with virial less than $-1/2$ actually exist. My thanks go to him for raising this question.

Dedication

This work is dedicated to my parents, Ricky Oneil Young and Betty Jo Norris Young, and my younger brother, Brett Joseph Young. Without their steadfast support, none of this would have been possible.

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Chapter 1

Introduction: An Overview of the Relativistic Vlasov-Poisson System

Consider a collection of N particles interacting through some force-field created by the particles themselves. Typically, the motion of the particles is governed by a set of coupled ODEs on phase space (say $\mathbb{R}^{6N}$ to make matters simple) of the form:

$$\begin{aligned} \dot{q}_i(t) &= v(p_i(t)) \\ \dot{p}_i(t) &= \sum_{j \neq i} F(q_j(t) - q_i(t)). \end{aligned}$$

Once some initial condition $(\mathring{p}_i, \mathring{q}_i)$ is chosen for each of the particles, the state of the system is completely determined for all future times (assuming some minimal regularity on the functions v and F).

The main problem with this description (from a practical point of view) is the sheer amount of information needed to specify the state of the system at any given time. In addition, it may be very difficult to develop any sort of intuitive idea about the behavior of the system from a collection of $6N$ numbers. The typical way to combat this issue is to replace the system of ODEs by a PDE for a single function that is expected to well-approximate the system under study - at least in the limit where the number of particles is large. That is, we replace a discrete dynamics by a fluid model. Even though we formally replace a finite-dimensional problem (the system of ODEs) by an infinite-dimensional problem (the PDE), the expectation is that we can expand the solution to the PDE in terms of certain basis functions and keep only the first few modes.

The essential questions to ask are:

1) Which PDE should we choose to approximate our system?

2) In what sense does the PDE approximate the original particle dynamics?

3) Can we give an idea of how closely our PDE approximated the particle dynamics (preferably in terms of some metric)?

The first question can be addressed easily in the situation where the overall force felt by any particular particle is not expected to grow with the number of particles. In this case, one usually takes a PDE whose characteristic curves match up with the trajectories of the particles. The heuristic idea is that a test particle placed in the fluid would follow the characteristics, and so the individual particles (which are assumed to be extremely small compared to the overall system) should more-or-less behave in the same fashion. The remaining questions can be more subtle (and are obviously related).

Take for instance a system of particles interacting according to

$$\begin{aligned} \dot{q}_i(t) &= p_i(t) \\ \dot{p}_i(t) &= \pm\frac{1}{N}\sum_{j\neq i}\frac{q_i(t)-q_j(t)}{|q_i(t)-q_j(t)|^3}, \end{aligned}$$

which is to say, a system of particles of mass $1/N$ interacting via Columbic repulsion (the "+" case) or Newtonian gravitation (the "–" case). The justification for setting the mass to $1/N$ is often left to the taste of the reader (or not given much explanation at all). Without this factor, it would be difficult to expect any limiting behavior for the system in general (as the force on any one particle could grow on the order of N). Whatever reasoning used to justify it, the presence of the factor $1/N$ in the dynamics is often referred to as the *mean field approximation*.

Of course for the attractive case, the singularity present in the force term above means that in general we only have existence of solutions for some finite time (as long as there are no coincident points in the initial data, the repulsive case has solutions for all times). To avoid such complications, we can regularize the force to smooth out the singularity. For the purposes of this heuristic discussion, we'll gloss over such issues. The corresponding PDE matching to this system of ODEs is the Vlasov-Poisson system ($\mathrm{VP}^{\pm}$):

$$\mathrm{VP}^{\pm}: \begin{cases} \left(\partial_t + p \cdot \nabla_q \pm \nabla_q \varphi_t(q) \cdot \nabla_p\right) f_t(p,q) = 0 \\ \\ \triangle_q \varphi_t(q) = 4\pi \int f_t(p,q)\, d^3p \\ \\ \varphi_t(q) \asymp -|q|^{-1} \text{ as } |q| \to \infty; \end{cases}$$

with repulsive Coulombic interaction (the "+" case) or attractive Newtonian interaction (the "–" case). The boundary condition at infinity merely serves to ensure that the system under study is effectively isolated from any external sources. Note that with this assumption we can formally write

$$\nabla_q \varphi_t(q) = \iint \frac{q - q'}{|q - q'|^3} f(p', q') d^3p' d^3q'$$

which recovers the analogous force term to the one from our dynamics.

As for mathematical studies of this PDE system, the first existence proof for (a regularized version) of $\mathrm{VP}^{\pm}$ was given by Batt in 1963. Around 1975, Neunzert succeeded in deriving a regularized version of $\mathrm{VP}^{\pm}$ from an underlying particle dynamics (regularized to avoid the singularities mentioned above). He also showed that classical solutions to this regularized form of $\mathrm{VP}^{\pm}$ are unique and exist for all times. Similar results were obtained (independently) by Braun and Hepp in 1977 and Dobrushin in 1978. See [Ne84] and [Sp91] for more details. In the late 1980s to early 1990s, Pfaffelmoser showed that classical solutions to the unregularized version of $\mathrm{VP}^{\pm}$ exist for all time [Pf92]. Until this result, most results about the unregularized system were restricted to local existence, or global existence for special kinds of initial data (highly symmetric data or data that were small in some way). See also [Rn97] for an alternate proof of the same theorem. More recently, Mouhot and Villani have shown that the Vlasov-Poisson system (at least in some perturbative regime) exhibits the unexpected phenomenon of Landau Damping. Landau himself discovered the phenomenon in 1946 only for a linearized version of the Vlasov-Poisson system. That the phenomenon is a feature of the nonlinear dynamics (and not an artefact of the linearization) was an outstanding problem until Mouhot and Villani's work in 2009. In brief, the reason for the excitement over this phenomenon is that it deceptively appears as a kind of irreversibility in a system whose equations are time reversible. See [Vi10] for a very readable account of this remarkable work.

We should mention that the naming of this PDE system is a little contentious. Historically, it seem that Jeans was the first to write down this equation - at least for the attractive case. He had in mind that it should be a natural candidate for a model of dilute stellar systems. Later, and it would seem independently of Jeans' work, Vlasov wrote down the Coulombic version of the PDEs in the context of studying multi-species plasmas [Vla38]. Predominantly, Vlasov's name has stuck in the literature (Poisson's name naturally coming from the Poisson equation appearing in the system). Other variants of the Vlasov-Poisson system have been proposed and studied. Vlasov himself also considered fully relativistic equations with electromagnetic interactions - the aptly named Vlasov-Maxwell (VM) system [Vla61]. As suggested by the name, the Vlasov-Maxwell system incorporates the Lorentz force into the PDE with the corresponding electric and magnetic fields produced by the flow of electricity itself (typically taken to be a multi-species plasma). Another system of note is the Vlasov-Einstein (VE) system. This system replaces the Newtonian gravitational force with that of Einstein's. While there are numerous results about these systems, their complexity makes a full accounting of them difficult (this is especially true for VE).

Of interest to us in this work is the *relativistic Vlasov-Poisson* (rVP) system which is given by (in units where $c = 1$)

$$\mathrm{rVP}^{\pm}: \begin{cases} \left(\partial_t + \frac{p}{\sqrt{1+|p|^2}} \cdot \nabla_q \pm \nabla_q \varphi_t(q) \cdot \nabla_p\right) f_t(p,q) = 0 \\ \triangle_q \varphi_t(q) = 4\pi \int f_t(p,q)\, d^3p \\ \varphi_t(q) \asymp -|q|^{-1} \text{ as } |q| \to \infty. \end{cases}$$

Similarly to the previous notation, rVP^+ models a single specie system with repulsive Coulombic interaction while rVP^- models a single specie system with attractive Newtonian interaction. This system is being studied in the literature as a simpler stepping stone to fully relativistic equations.

One of the earliest papers to appear on this system is [GS85] wherein Glassey and Schaeffer show that global classical solutions to $\mathrm{rVP}^{\pm}$ will exist for initial data that are spherically symmetric, compactly supported in momentum space, and vanish on characteristics with vanishing angular momentum (hereafter abbreviated by I.D.) which are in addition compactly supported in $\mathbb{R}^6$ and have $\mathfrak{L}^{\infty}$-norm below a critical constant $\mathcal{C}^{\pm}_{\infty}$, with $\mathcal{C}^{+}_{\infty} = \infty$ and $\mathcal{C}^{-}_{\infty} < \infty$. More recently, Kiessling and Tahvildar-Zadeh [KTZ08] have extended the theorem of Glassey and Schaeffer for rVP^- by proving global existence of classical solutions for I.D. which are in $\mathfrak{P}_1 \cap \mathfrak{C}^1$ † and have $\mathfrak{L}^{\beta}$-norm below a critical constant $\mathcal{C}^{-}_{\beta}$ with $\mathcal{C}^{-}_{\beta} < \infty$, and $\mathcal{C}^{-}_{\beta}$ identically zero iff $\beta < 3/2$. The authors explicitly computed $\mathcal{C}^{-}_{3/2}$ but characterized the constant for other values of $\beta > 3/2$ as a variational problem. In this work, we will solve this variational problem and so determine the sharp values of $\mathcal{C}^{-}_{\beta}$ for all $\beta \geq 3/2$.

Glassey and Schaeffer also investigated what may happen when rVP^- is launched by initial data with $\|f\|_{\infty} > \mathcal{C}^{-}_{\infty}$. They proved that negative energy data lead to "blow-up" (i.e. formation of a singularity) in finite time. This is in sharp contradistinction to the non-relativistic Vlasov Poisson system with attractive coupling (VP^-) which does not exhibit finite time blow-up for classical data. Indeed, the possibility of collapse

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† $\mathfrak{P}_n \cap \mathfrak{C}^k$ is the set of probability measures on $\mathbb{R}^6$ absolutely continuous w.r.t. Lebesgue measure whose first n moments are finite and whose Radon-Nikodym derivative is $\mathfrak{C}^k$.

for solutions to rVP^- is a primary motivation for studying the system - as the collapse is due to "relativistic effects." In [LMR08b], Lemou, Méhats, and Raphaël proved that systems launched by initial data with negative total energy approach self-similar collapse. Around the same time, Kiessling and Tahvildar-Zadeh proved that any spherically symmetric classical solution of rVP^- launched by I.D. satisfying $f_0 \in \mathfrak{P}_3 \cap \mathfrak{C}^1$ with *zero total energy* and total virial less than or equal to $-1/2$ will blow up in finite time (Theorem 6.1 of [KTZ08]). However, they left open the question whether such I.D. existed. The existence of such I.D. is answered affirmatively in Chapter 3 of this work.

We should mention two related lines of research which will not, however, be pursued in this work. These concern the nonlinear stability of stationary solutions of rVP^- and the dynamical details of the solutions which blow-up in finite time. Such questions have been pursued in [HR07] and in [LMR08a, LMR09]. Namely, Hadžić and Rein [HR07] showed the non-linear stability of a wide class of steady-state solutions of rVP^- against certain allowable perturbations utilizing energy-Casimir functionals. Shortly thereafter, Lemou, Méhats, and Raphaël [LMR08a, LMR09] investigated non-linear stability versus the formation of singularities in rVP^- through concentration compactness techniques.

Despite many years of interest in rVP^-, not much work has focused on the derivation of this continuum model from an underlying particle model in the appropriate limit. Indeed, beyond Neunzert's work on the regularized Vlasov-Poisson system in the mid 1970s, comparatively little attention has been given to establishing any of the other Vlasov models (rVP, VM, or VE) as an infinite particle limit of an underlying particle dynamics. Work by Elskens, Kiessling, and Ricci [EKR09] has extended these ideas to relativistic particles interacting via scalar wave fields (with non-relativistic smoothing to remove singularities). These authors have recently extended their results to a derivation of the relativistic Vlasov-Maxwell system from (non-relativistically smoothed) point-particle dynamics (publication in preparation). In Chapter 4 of this thesis we address the derivation of rVP^- from some suitable N-body dynamics.

Our simplest option is to adapt Neunzert's original proof by considering particles with relativistic velocities interacting by Newtonian gravity (which we shall detail for purposes of review). However, this underlying point-particle dynamics is physically incorrect! Relativistically moving particles interacting gravitationally must be modeled using general relativity. The Vlasov limit of such a general relativistic N-body system is expected to yield the general relativistic Vlasov-Einstein equations. We would expect rVP$^-$ to result from a *weak-field* limit of the Vlasov-Einstein system. Currently, the only work known to the author is [Re94] wherein Rendall proves that sufficiently regular, asymptotically flat initial data for the Vlasov-Einstein system launches solutions which are well approximated by the *non-relativistic* Vlasov-Poisson system (VP$^-$). Even if rVP$^-$ is found to be the limiting behavior in some sense of a weak-field regime for Vlasov-Einstein, many of the interesting results about rVP$^-$ (finite time blow-up, for instance) are in the *strong-field* regime. This might lead one to suspect that rVP$^-$ is a mathematical hybrid without much physical basis.

Yet, in [KTZ08], Kiessling and Tahvildar-Zadeh proposed that rVP$^-$ might obtain as the Vlasov Limit of a two-specie charged plasma on an alternate space-time scale compared to the usual one governed by the conventional Vlasov Limit. In a nutshell, the authors envision an overall neutral two-specie charged plasma (where the species have equal mass but opposite charge) initially distributed identically, independently, and spherically symmetrically which interact via regularized Maxwell fields (to avoid the singularities plaguing electromagnetic self-interactions). As long as the distribution remains spherically symmetric, the magnetic contribution to the interaction should be negligible. This leaves the electrostatic interaction amongst the particles. Each particle then experiences an overall attractive force directed approximately toward the center of the distribution because each "+" charge sees the remaining system as having an overall extra "–" charge (and vice versa, of course) as the other charges roughly cancel. Hence, we should get rVP$^-$ with perfectly central force field in the limit as the number of particles goes to infinity *on appropriate space-time scales*. Interestingly, this derivation *does not* depend on being in a weak-field regime! The main part of Chapter 4 of this thesis is devoted to a detailed examination of this proposal.

To summarize, in Chapter 2 we compute the sharp value of the constant $\mathcal{C}_\beta^-$ for $\beta \geq 3/2$ from [KTZ08]. In Chapter 3, we settle the question of whether there exist I.D.

satisfying $f_0 \in \mathfrak{P}_3 \cap \mathfrak{C}^1$ with *zero total energy* and total virial less than or equal to $-1/2$ by constructing a family of examples. Finally in Chapter 4, we examine the proposal for the derivation of rVP$^-$ given in [KTZ08] for a simpler model with only Coulombic interactions. We will find clear evidence for the relevance of *both* an rVP$^-$-type force term and a dissipative operator that govern the dynamics. Even though this model uses highly simplified physics for the underlying particle dynamics, we persevere in the hope that our findings will be vindicated by a future study of the full electromagnetic model with spherical symmetry.

Chapter 2

Optimal $\mathfrak{L}^\beta$-Control for the Global Cauchy Problem of the Relativistic Vlasov-Poisson System

2.1 Introduction

As mentioned in the introduction, the relativistic Vlasov-Poisson (rVP) system is given by

$$\text{rVP}^{\pm}: \begin{cases} \left(\partial_t + \frac{p}{\sqrt{1+|p|^2}} \cdot \nabla_q \pm \nabla_q \varphi_t(q) \cdot \nabla_p\right) f_t(p,q) = 0 \\ \triangle_q \varphi_t(q) = 4\pi \int f_t(p,q)\, d^3p \\ \varphi_t(q) \asymp -|q|^{-1} \text{ as } |q| \to \infty; \end{cases}$$

rVP^+ models a system with repulsive interaction while rVP^- models a system with attractive interaction.

In this work, we focus exclusively on the attractive case and henceforth suppress the superscript on both rVP$^-$. In [KTZ08], Kiessling and Tahvildar-Zadeh prove that a unique global classical solution to the relativistic Vlasov-Poisson system exists whenever the positive, integrable initial datum f_0 is spherically symmetric, compactly supported in momentum space, vanishes on characteristics with vanishing angular momentum, and has $\mathfrak{L}^\beta$-norm below a critical constant $\mathcal{C}_\beta$, with $\mathcal{C}_\beta > 0$ if and only if $\beta \geq 3/2$. The constant $\mathcal{C}_\beta$ is critical in the sense that, everything else being equal, initial data can be found which lead to blow-up in finite time if their $\mathfrak{L}^\beta$-norm is allowed to be ever so slightly bigger than $\mathcal{C}_\beta$. This critical constant is given by the following minimization problem:

$$\mathcal{C}_\beta \equiv \inf_{\mathfrak{P}_1 \cap \mathfrak{L}^\beta} \Phi_\beta(f), \tag{2.1}$$

$$\Phi_\beta(f) \equiv \left(\frac{\mathcal{E}_p^u(f)}{-\mathcal{E}_q(f)}\right)^{3(1-\frac{1}{\beta})} \|f\|_\beta, \tag{2.2}$$

where $\mathfrak{P}_1 \cap \mathfrak{L}^\beta$ denotes the set of probability measures on $\mathbb{R}^6$ with finite first moment which are absolutely continuous with respect to Lebesgue measure having density in $\mathfrak{L}^\beta$. The functionals $\mathcal{E}_p^u$ and $\mathcal{E}_q$ are given by

$$\mathcal{E}_p^u(f) \equiv \iint |p| f(p,q)\, d^3p\, d^3q, \tag{2.3}$$

$$\mathcal{E}_q(f) \equiv -\frac{1}{2} \iiiint \frac{f(p',q')f(p,q)}{|q-q'|}\, d^3p'\, d^3p\, d^3q'\, d^3q. \tag{2.4}$$

It is further shown that

$$\left[\left(\frac{3}{8}\right)^3 \frac{15}{16}\right]^{1-\frac{1}{\beta}} \leq \mathcal{C}_\beta \leq \frac{45}{8\pi^2}\left(\frac{8\pi^{\frac{5}{2}}}{\prod_{k=1}^3 (k+2\beta)} \frac{\Gamma(\beta)}{\Gamma\left(\beta+\frac{3}{2}\right)}\right)^{\frac{1}{\beta}}. \tag{2.5}$$

In particular, the optimal constant for $\beta = \frac{3}{2}$ is explicitly calculated in the paper. Its value is $\frac{3}{8}\left(\frac{15}{16}\right)^{\frac{1}{3}}$, which is equal to the one given by either bound.

In this chapter, we address the determination of the critical norm for all remaining cases. We begin in the next section by proving the existence of minimizers for Φ_β in a slightly larger class of functions Ω_β to be defined below. After that, we characterize the minimizers via variational techniques and find that they are the well known Lane-Emden polytropes. This will show that the minimizers are actually in our original space $\mathfrak{P}_1 \cap \mathfrak{L}^\beta$. Finally, we compute the optimal constant in terms of the parameter β and the first zero and corresponding slope of the standard polytropes. At the end, we mention some numerical results pertaining to the calculation of $\mathcal{C}_\beta$.

NOTE: The results of this chapter have been refereed and accepted for publication in the journal *Transport Theory and Statistical Physics.*

2.2 Existence of Minimizers

To begin with, as remarked in the "note added" in [KTZ08], the variational problem (1),(2) is equivalent after rescaling to the one given by Lemou, Méhats, and Raphaël in [LMR08b], who designed it for different purposes, namely to study blow up dynamics for the relativistic Vlasov-Poisson system. Instead of giving a detailed analysis, they refer the reader to their earlier work [LMR08a] where they study an analogous variational principle (given in formula 1.18) for the non-relativistic Vlasov-Poisson system. Thus, in principle we could build on their results to evaluate C_β. Instead, here we first give a somewhat different proof of the existence and characterization of minimizers by combining the techniques of Weinstein ([W83]) (also referred to by Lemou, Méhats, and Raphaël) with those of Lieb and Simon ([LS77]). This strategy has certain advantages, as we shall see in the next section.

We find it convenient (for reasons explained below) to expand the class of functions over which we attempt to minimize Φ_β. To that end, define

$$\Omega_\beta = \{f : \mathbb{R}^6 \to \mathbb{R} : f \geq 0, \|f\|_1 + \||p|f\|_1 + \|f\|_\beta < \infty\}$$

and note that $\mathfrak{P}_1 \cap \mathfrak{L}^\beta \subset \Omega_\beta$ (indeed, a function $f \in \Omega_\beta$ will also be in $\mathfrak{P}_1 \cap \mathfrak{L}^\beta$ whenever $\||q|f\|_1$ is finite and the $\mathfrak{L}^1$-norm is equal to 1). Since we have allowed functions of arbitrary $\mathfrak{L}^1$-norm into our considerations, we need to adjust our definition of Φ_β. Assuming $\|f\|_1 > 0$ and inserting $\frac{f}{\|f\|_1}$ into Φ_β, we arrive at the appropriate functional:

$$\widetilde{\Phi}_\beta(f) \equiv \left(\frac{\mathcal{E}_p^u(f)}{-\mathcal{E}_q(f)}\right)^{3(1-\frac{1}{\beta})} \|f\|_\beta \, \|f\|_1^{2-\frac{3}{\beta}}. \tag{2.6}$$

We now seek to minimize $\widetilde{\Phi}_\beta$ over this enlarged space of functions. In the next section, we will show that there exist minimizers for $\widetilde{\Phi}_\beta$ over Ω_β that are also in $\mathfrak{P}_1 \cap \mathfrak{L}^\beta$.

The demonstration that minimizers exist closely follows Weinstein ([W83]). Since $\widetilde{\Phi}_\beta(f) \geq 0$ for all functions in Ω_β, we can find a minimizing sequence $\{f_{\beta,n}\} \subset \Omega_\beta$ so that

$$\widetilde{\mathcal{C}}_\beta \equiv \inf_{f\in\Omega_\beta} \widetilde{\Phi}_\beta(f) = \lim_{n\to\infty} \widetilde{\Phi}_\beta(f_{\beta,n}). \tag{2.7}$$

Since we are minimizing over a larger class of functions, $\widetilde{\mathcal{C}}_\beta \leq \mathcal{C}_\beta$ and so the upper bound noted above still holds. Applying the well-known procedure of spherically symmetric equi-measurable rearrangements (c.f. [LL01, Chapter 3]), we can assume $f_{\beta,n}(p,q) = f_{\beta,n}(|p|,|q|,\theta)$ (committing a slight abuse of notation) where θ is the angle between p and q.

For any positive real numbers κ, λ and μ, the triple family of scaling

$$f_{\kappa,\lambda,\mu}(p,q) \equiv \mu\, f(\lambda p, \kappa q) \tag{2.8}$$

leaves $\widetilde{\Phi}_\beta$ invariant while

$$\mathcal{E}_p^u(f_{\kappa,\lambda,\mu}) = \frac{\mu}{\lambda^4\kappa^3}\mathcal{E}_p^u(f), \tag{2.9}$$

$$\mathcal{E}_q(f_{\kappa,\lambda,\mu}) = \frac{\mu^2}{\lambda^6\kappa^5}\mathcal{E}_q(f), \tag{2.10}$$

$$\|f_{\kappa,\lambda,\mu}\|_r = \frac{\mu}{(\kappa\lambda)^{\frac{3}{r}}}\|f\|_r. \tag{2.11}$$

Taking advantage of this scaling invariance, we can assume that our minimizing sequence has the following properties:

$$f_{\beta,n}(p,q) = f_{\beta,n}(|p|,|q|,\theta), \tag{2.12}$$

$$\mathcal{E}_p^u(f_{\beta,n}) = 1, \tag{2.13}$$

$$\|f_{\beta,n}\|_1 = 1, \tag{2.14}$$

$$\|f_{\beta,n}\|_\beta = 1. \tag{2.15}$$

Hence, the Banach-Alaoglu theorem and the reflexivity of $\mathfrak{L}^\beta$ for $\beta \in (1,\infty)$ give us a function $f_\beta \in \mathfrak{L}^\beta$ such that some subsequence of $\{f_{\beta,n}\}_{n=1}^\infty$ converges weakly in $\mathfrak{L}^\beta$ to f_β. Without loss of generality, we assume that we have already extracted this subsequence. Standard arguments concerning lower semi-continuity and weak convergence (c.f. [FL07]) show that

$$\mathcal{E}_p^u(f_\beta) \leq 1, \tag{2.16}$$

$$\|f_\beta\|_1 \leq 1, \tag{2.17}$$

$$\|f_\beta\|_\beta \leq 1, \tag{2.18}$$

so that $f_\beta \in \Omega_\beta$ (this is the advantage of expanding the space of functions as showing that $f_\beta \in \mathfrak{P}_1 \cap \mathfrak{L}^\beta$ is difficult at this point). Thus, we can conclude the following is true:

$$\widetilde{\mathcal{C}}_\beta \leq \widetilde{\Phi}_\beta(f_\beta) = \left(\frac{\mathcal{E}_p^u(f_\beta)}{-\mathcal{E}_q(f_\beta)}\right)^{3(1-\frac{1}{\beta})} \|f_\beta\|_\beta \, \|f_\beta\|_1^{2-\frac{3}{\beta}} \leq (-\mathcal{E}_q(f_\beta))^{-3(1-\frac{1}{\beta})}. \tag{2.19}$$

Since by construction

$$\lim_{n\to\infty} \widetilde{\Phi}_\beta(f_{\beta,n}) = \lim_{n\to\infty} (-\mathcal{E}_q(f_{\beta,n}))^{-3(1-\frac{1}{\beta})} = \widetilde{\mathcal{C}}_\beta, \tag{2.20}$$

we will be done if we can show that $\mathcal{E}_q(f_{\beta,n})$ converges to $\mathcal{E}_q(f_\beta)$. Unfortunately, $\mathcal{E}_q$ is upper semi-continuous with respect to weak convergence, and so all we can immediately conclude is that

$$\mathcal{E}_q(f_\beta) \geq \lim_{n\to\infty} \mathcal{E}_q(f_{\beta,n}). \tag{2.21}$$

To show the convergence of $\mathcal{E}_q(f_{\beta,n})$ to $\mathcal{E}_q(f_\beta)$, we first rewrite the potential energy functional as

$$\mathcal{E}_q(f) = -\frac{1}{2}\int \rho_f(q)\, K_f(q)\, d^3q, \tag{2.22}$$

where

$$\rho_f(q) \equiv \int f(p,q)\, d^3p, \tag{2.23}$$

and

$$K_f(q) \equiv \rho_f * |\mathrm{Id}|^{-1}(q) = \int \frac{\rho_f(q')}{|q-q'|}\, d^3q'. \tag{2.24}$$

We seek bounds on $\rho_{f_{\beta,n}}$ which will then imply bounds on $K_{f_{\beta,n}}$. To that end, Lemma 4.3 in [KTZ08] implies that

$$\|\rho_f\|_{\gamma_\beta} \leq C(\beta)\|f\|_\beta^{\eta_\beta}\, \mathcal{E}_p^u(f)^{1-\eta_\beta} \tag{2.25}$$

with exponents given by

$$\gamma_\beta \equiv \frac{4\beta - 3}{3\beta - 2} \quad \text{and} \quad \eta_\beta \equiv \frac{\beta}{4\beta - 3}. \tag{2.26}$$

Note that γ_β is an increasing function of β and that the limiting case of $\beta = \frac{3}{2}$ gives $\gamma_{\frac{3}{2}} = \frac{6}{5}$. Thus ρ_{f_β} and $\rho_{f_{\beta,n}}$ are in $\mathfrak{L}^1(\mathbb{R}^3) \cap \mathfrak{L}^{\gamma_\beta}(\mathbb{R}^3)$ for all β and all n. Also note that the sequence $\{\rho_{f_{\beta,n}}\}_{n=1}^\infty$ is uniformly bounded in $\mathfrak{L}^{\gamma_\beta}$-norm, and so some subsequence must converge weakly in this space. Standard arguments show that this weak limit must equal ρ_{f_β} a.e.

Next, we can conclude that $K_{f_{\beta,n}}$ and K_{f_β} are in $\mathfrak{L}^\alpha_{\text{loc}}(\mathbb{R}^3)$ for $3 \le \alpha \le \frac{12\beta-9}{\beta}$ ([LL01, Theorem 10.2]). Note that $\frac{12\beta-9}{\beta}$ is also an increasing function of β and the limiting case of $\beta = \frac{3}{2}$ makes this exponent equal to 6. We can turn the local estimates into global ones for all $\alpha > 3$ via the following growth estimate on the potential.

For any spherically symmetric $f(p,q)$, the marginal mass distribution $\rho_f(q)$ will also be spherically symmetric. Hence, the well known formula for the potential of a spherically symmetric mass distribution gives

$$K_f(|q|) = \frac{4\pi}{|q|}\int_0^{|q|} \rho_f(r)\, r^2\, dr + 4\pi \int_{|q|}^\infty \rho_f(r)\, r\, dr \tag{2.27}$$

$$\le \frac{4\pi}{|q|}\int_0^{|q|} \rho_f(r)\, r^2\, dr + \frac{4\pi}{|q|} \int_{|q|}^\infty \rho_f(r)\, r^2\, dr \tag{2.28}$$

$$\le \frac{1}{|q|}, \tag{2.29}$$

where in the last step we have used that for our purposes $\|\rho_f\|_1 \le 1$. So, if $K_f \in \mathfrak{L}^\alpha_{\text{loc}}(\mathbb{R}^3)$ for $\alpha > 3$, then we have for any $R > 0$

$$\int K_f^\alpha(q)\, d^3q = \int_{|q|<R} K_f^\alpha(q)\, d^3q + \int_{|q|\ge R} K_f^\alpha(q)\, d^3q \tag{2.30}$$

$$\le \int_{|q|<R} K_f^\alpha(q)\, d^3q + \frac{4\pi}{(\alpha-3)R^{\alpha-3}}. \tag{2.31}$$

Thus, we may conclude that for the distributions we are considering, $K_{f_{\beta,n}}$ and K_{f_β} are in $\mathfrak{L}^\alpha(\mathbb{R}^3)$ for $3 < \alpha \le \frac{12\beta-9}{\beta}$. Note that this proves that $\rho_f K_f$ is indeed in $\mathfrak{L}^1(\mathbb{R}^3)$ for spherically symmetric $f \in \Omega_\beta$.

As with $\rho_{f_{\beta,n}}$, we have a bound on the $\mathfrak{L}^\alpha$-norm of $K_{f_{\beta,n}}$ that is independent of n, and so some subsequence of $\{K_{f_{\beta,n}}\}_{n=1}^\infty$ must converge weakly in this space. As usual, this weak limit must equal K_{f_β} a.e.

We can actually say something much stronger about the convergence of $\{K_{f_{\beta,n}}\}$ to K_{f_β} - namely that any subsequence converging weakly to K_{f_β} will actually converge strongly in $\mathfrak{L}^r$ on sets of finite measure for any $r < \frac{12\beta-9}{\beta}$. To see this, we note that since $\nabla \cdot \nabla K_{f_{\beta,n}}$ is in $\mathfrak{L}^1 \cap \mathfrak{L}^{\gamma_\beta}$, we know that $\nabla K_{f_{\beta,n}}$ is locally in $\mathfrak{L}^{\frac{3}{2}} \cap \mathfrak{L}^{\kappa_\beta}$ where $\kappa_\beta = \frac{12\beta-9}{5\beta-3}$. Thus, $\nabla K_{f_{\beta,n}}$ is locally in $\mathfrak{L}^2$ whenever $\beta > \frac{3}{2}$. Arguments like the one leading to (2.29) for $K_{f_{\beta,n}}$ show that these functions decay rapidly enough at infinity to be in $\mathfrak{L}^2$ proper, and that the decay is independent of n. Again, some subsequence of $\{\nabla K_{f_{\beta,n}}\}$ will converge weakly in $\mathfrak{L}^2$ and must converge to ∇K_{f_β}. Following the proof of [LL01, Theorem 8.6], we have the strong convergence stated above.

Combining our upper bound for K_f and the local strong convergence gives us that $\{K_{f_{\beta,n}}\}$ converges strongly to K_{f_β} in $\mathfrak{L}^\alpha$ for $3 < \alpha < \frac{12\beta-9}{\beta}$:

$$\|K_{f_{\beta,n}} - K_{f_\beta}\|_\alpha^\alpha = \|\left(K_{f_{\beta,n}} - K_{f_\beta}\right)\chi_{B_R(0)}\|_\alpha^\alpha + \int_{B_R^c(0)} |K_{f_{\beta,n}} - K_{f_\beta}|^\alpha d^3q \tag{2.32}$$

$$\leq \|\left(K_{f_{\beta,n}} - K_{f_\beta}\right)\chi_{B_R(0)}\|_\alpha^\alpha + \int_{B_R^c(0)} \frac{1}{|q|^\alpha}\, d^3q \tag{2.33}$$

$$\leq \|\left(K_{f_{\beta,n}} - K_{f_\beta}\right)\chi_{B_R(0)}\|_\alpha^\alpha + \frac{4\pi}{(\alpha-3)R^{\alpha-3}}. \tag{2.34}$$

Finally, we are in a position to show the convergence of $\mathcal{E}_q(f_{\beta,n})$ to $\mathcal{E}_q(f_\beta)$. We assume that $\beta > \frac{3}{2}$ and hence $\rho_f \in \mathfrak{L}^{\frac{6}{5}}$ and $K_f \in \mathfrak{L}^6$.

$$2|\mathcal{E}_q(f_\beta) - \mathcal{E}_q(f_{\beta,n})| = \left|\int \rho_{f_{\beta,n}}(q)K_{f_{\beta,n}}(q) - \rho_{f_\beta}(q)K_{f_\beta}(q)d^3q\right| \tag{2.35}$$

$$\leq \left|\int \rho_{f_{\beta,n}}(q)(K_{f_{\beta,n}}(q) - K_{f_\beta}(q))d^3q\right| + \left|\int (\rho_{f_{\beta,n}}(q) - \rho_{f_\beta}(q))K_{f_\beta}(q)d^3q\right| \tag{2.36}$$

$$\leq \|\rho_{f_{\beta,n}}\|_{\frac{6}{5}}\|K_{f_{\beta,n}} - K_{f_\beta}\|_6 + \left|\int (\rho_{f_{\beta,n}}(q) - \rho_{f_\beta}(q))K_{f_\beta}(q)d^3q\right|. \tag{2.37}$$

The first term in the last inequality can be made arbitrarily small since $K_{f_{\beta,n}}$ converges strongly to K_{f_β} in $\mathfrak{L}^6$ for $\beta > \frac{3}{2}$ and the $\mathfrak{L}^{\frac{6}{5}}$-norm of $\rho_{f_{\beta,n}}$ is bounded above independently of n by standard interpolation estimates. The second term can be made arbitrarily small by the weak convergence of $\rho_{f_{\beta,n}}$ to ρ_{f_β} in $\mathfrak{L}^{\frac{6}{5}}$ (after possibly another subsequence extraction) and the fact that K_{f_β} is in the dual space - $\mathfrak{L}^6$.

We see that inequality (2.19) is saturated, and we have that

$$\widetilde{\mathcal{C}}_\beta = \widetilde{\Phi}_\beta(f_\beta). \tag{2.38}$$

Note that this forces

$$\mathcal{E}_p^u(f_\beta) = 1, \tag{2.39}$$

$$\|f_\beta\|_1 = 1, \tag{2.40}$$

$$\|f_\beta\|_\beta = 1. \tag{2.41}$$

Thus, we have shown the existence of minimizers for $\widetilde{\Phi}_\beta$ over the space Ω_β satisfying the properties given above.

In the next section, we show that f_β can be chosen so that it is compactly supported for $\beta > \frac{3}{2}$, so that among all possible minimizers for $\widetilde{\Phi}_\beta$ there is indeed one in $\mathfrak{P}_1 \cap \mathfrak{L}^\beta$ - proving that $\widetilde{\mathcal{C}}_\beta = \mathcal{C}_\beta$.

2.3 Identification of Minimizers

We want to find the infimum of $\widetilde{\Phi}_\beta$ over the space of functions Ω_β introduced in the last section. Following an idea in Lieb-Simon ([LS77]) we first note that Ω_β is a convex space of functions so that, if f_β is a minimizer of our functional over Ω_β and η is any function in this space, then for all $0 \leq t \leq 1$ we have that $(1-t)f_\beta + t\eta \in \Omega_\beta$. Consequently, we can consider

$$\left.\frac{d}{dt}\right|_{t=0^+} \widetilde{\Phi}_\beta((1-t)f_\beta + t\eta), \tag{2.42}$$

where by $t = 0^+$ we have in mind the one-sided Gateaux derivative from the right at zero. This technique avoids the difficulty that arbitrary variations of a given $f \in \Omega_\beta$ may become negative (and hence no longer belong to Ω_β). Direct calculation gives us that this derivative is

$$\widetilde{\Phi}_\beta(f_\beta)\left\{\left(3-\tfrac{3}{\beta}\right)\left(\frac{\left.\frac{d}{dt}\right|_{t=0^+}\mathcal{E}_p^u((1-t)f_\beta+t\eta)}{\mathcal{E}_p^u(f_\beta)} - \frac{\left.\frac{d}{dt}\right|_{t=0^+}\mathcal{E}_q((1-t)f_\beta+t\eta)}{\mathcal{E}_q(f_\beta)}\right)\right.$$
$$\left. + \frac{\left.\frac{d}{dt}\right|_{t=0^+}\|(1-t)f_\beta+t\eta\|_\beta}{\|f_\beta\|_\beta} + (2-\tfrac{3}{\beta})\frac{\left.\frac{d}{dt}\right|_{t=0^+}\|(1-t)f_\beta+t\eta\|_1}{\|f_\beta\|_1}\right\}. \tag{2.43}$$

We now compute the indicated derivatives separately. We begin with the ultra-relativistic kinetic energy:

$$\left.\frac{d}{dt}\right|_{t=0^+} \mathcal{E}_p^u((1-t)f_\beta + t\eta) \quad = \quad -\mathcal{E}_p^u(f_\beta) + \iint |p| \; \eta(p,q) d^3p d^3q. \tag{2.44}$$

Next, we easily compute:

$$\left.\frac{d}{dt}\right|_{t=0^+} \|(1-t)f_\beta + t\eta\|_r = \\ \|f_\beta\|_r^{1-r} \left(-\|f_\beta\|_r^r + \iint \left(f_\beta(p,q)\right)^{r-1} \; \eta(p,q) \; d^3p \, d^3q \right). \tag{2.45}$$

Finally, we find:

$$\left.\frac{d}{dt}\right|_{t=0^+} \mathcal{E}_q((1-t)f_\beta + t\eta) \quad = \quad -2\left(\mathcal{E}_q(f_\beta)\right) - \iint K_\beta(q)\eta(p,q) d^3p d^3q \tag{2.46}$$

where we have used K_β as in the previous section. Inserting these into our derivative above, collecting terms and noting that all the constant terms cancel (i.e. those terms not involving an integration against η) yields

$$\left.\frac{d}{dt}\right|_{t=0^+} \widetilde{\Phi}_\beta((1-t)f_\beta + t\eta) = \tag{2.47}$$
$$\iint \left[\left(3 - \frac{3}{\beta}\right) \left(\frac{|p|}{\mathcal{E}_p^u(f_\beta)} - \frac{K_\beta(q)}{-\mathcal{E}_q(f_\beta)} \right) + \frac{(f_\beta(p,q))^{\beta-1}}{\|f_\beta\|_\beta^\beta} + \frac{2-\frac{3}{\beta}}{\|f_\beta\|_1} \right] \eta(p,q) \; d^3p \; d^3q.$$

Since the indicator function of any set of finite measure is in Ω_β, we are tempted to simply conclude that

$$\left(3 - \frac{3}{\beta}\right) \left(\frac{|p|}{\mathcal{E}_p^u(f_\beta)} - \frac{K_\beta(q)}{-\mathcal{E}_q(f_\beta)} \right) + \frac{(f_\beta(p,q))^{\beta-1}}{\|f_\beta\|_\beta^\beta} + \frac{2-\frac{3}{\beta}}{\|f_\beta\|_1} \equiv 0 \tag{2.48}$$

or equivalently that

$$f_\beta(p,q) = \|f_\beta\|_\beta^{\frac{\beta}{\beta-1}} \left(\left(3 - \frac{3}{\beta}\right) \left(\frac{K_\beta(q)}{-\mathcal{E}_q(f_\beta)} - \frac{|p|}{\mathcal{E}_p^u(f_\beta)} \right) - \frac{2-\frac{3}{\beta}}{\|f_\beta\|_1} \right)^{\frac{1}{\beta-1}}. \tag{2.49}$$

Such a function cannot be in Ω_β since whenever

$$\left(3 - \frac{3}{\beta}\right) \left(\frac{K_\beta(q)}{-\mathcal{E}_q(f_\beta)} - \frac{|p|}{\mathcal{E}_p^u(f_\beta)} \right) - \frac{2-\frac{3}{\beta}}{\|f_\beta\|_1} < 0, \tag{2.50}$$

we get complex values for f_β in general. Hence, we take

$$f_\beta(p,q) \quad \equiv \quad \|f_\beta\|_\beta^{\frac{\beta}{\beta-1}} \left(\left(3 - \frac{3}{\beta}\right) \left(\frac{K_\beta(q)}{-\mathcal{E}_q(f_\beta)} - \frac{|p|}{\mathcal{E}_p^u(f_\beta)} \right) - \frac{2-\frac{3}{\beta}}{\|f_\beta\|_1} \right)_+^{\frac{1}{\beta-1}}, \tag{2.51}$$

where $(\cdot)_+$ means the positive part of the argument.

Since we have altered the natural minimizer so that it lies in Ω_β, we need to examine the effect on the one-sided Gateaux derivatives of our functional. Let Λ be the support of our minimizer f_β. Every $\eta \in \Omega_\beta$ can be decomposed as $\eta = \eta\chi_\Lambda + \eta\chi_{\Lambda^c}$. Hence,

$$\left.\frac{d}{dt}\right|_{t=0^+} \widetilde{\Phi}_\beta((1-t)f_\beta + t\eta) = \tag{2.52}$$
$$\iint_{\Lambda^c} \left[\left(3 - \frac{3}{\beta}\right) \left(\frac{|p|}{\mathcal{E}_p^u(f_\beta)} - \frac{K_\beta(q)}{-\mathcal{E}_q(f_\beta)} \right) + \frac{(f_\beta(p,q))^{\beta-1}}{\|f_\beta\|_\beta^\beta} + \frac{2 - \frac{3}{\beta}}{\|f_\beta\|_1} \right] \eta(p,q)\; d^3p\; d^3q.$$

But on the set Λ^c, we have that $f_\beta \equiv 0$ and that the integrand is strictly positive. Hence, for any $\eta \in \Omega_\beta$

$$\left.\frac{d}{dt}\right|_{t=0^+} \widetilde{\Phi}_\beta((1-t)f_\beta + t\eta) \geq 0, \tag{2.53}$$

showing that f_β as defined in (2.51) is indeed a minimizer for $\widetilde{\Phi}_\beta$ over Ω_β.

We now use scaling invariance to pick out a special minimizer that we will use to calculate $\widetilde{\mathcal{C}}_\beta$. To that end, recall that the triple family of scalings as given in (2.8)

$$f_{\kappa,\lambda,\mu}(p,q) \equiv \mu\; f(\lambda p, \kappa q)$$

(where κ, λ and μ are positive real numbers) leaves $\widetilde{\Phi}_\beta$ invariant. As noted in the previous section, using this scaling we can choose f_β so that its $\mathfrak{L}^1$-norm is 1. We break from the remaining choices above by requiring $-\mathcal{E}_q(f_\beta) = \mathcal{E}_p^u(f_\beta)$ (as opposed to $\mathcal{E}_p^u(f_\beta) = 1$). Hence, we can no longer choose the $\mathfrak{L}^\beta$-norm to be 1. Such choices are completely arbitrary since we have an infinite number of minimizers from which to pick. However, these choices give a minimizer of particularly nice form.

Our choices so far force

$$\kappa\lambda = \|f_\beta\|_1 \frac{\mathcal{E}_p^u(f_\beta)}{-\mathcal{E}_q(f_\beta)}, \tag{2.54}$$

and

$$\mu = \frac{(\kappa\lambda)^3}{\|f_\beta\|_1} = \|f_\beta\|_1^2 \left(\frac{\mathcal{E}_p^u(f_\beta)}{-\mathcal{E}_q(f_\beta)} \right)^3, \tag{2.55}$$

which leaves us some room to choose either κ or λ in a convenient way. Using (2.51), we see that

$$f_{\beta;\,\kappa,\lambda,\mu}(p,q)=\mu\left(\left(3-\frac{3}{\beta}\right)\|f_\beta\|_\beta^\beta\left(\frac{K_\beta(\kappa q)}{-\mathcal{E}_q(f_\beta)}-\frac{|\lambda p|}{\mathcal{E}_p^u(f_\beta)}\right)-\frac{\left(2-\frac{3}{\beta}\right)\|f_\beta\|_\beta^\beta}{\|f_\beta\|_1}\right)_+^{\frac{1}{\beta-1}} \tag{2.56}$$

$$=\mu\left(\left(3-\frac{3}{\beta}\right)\|f_\beta\|_\beta^\beta\left(\frac{\|f_\beta\|_1}{-\mathcal{E}_q(f_\beta)}\frac{1}{\kappa}K_\beta^{\kappa,\lambda,\mu}(q)-\frac{\lambda}{\mathcal{E}_p^u(f_\beta)}|p|\right)-\frac{\left(2-\frac{3}{\beta}\right)\|f_\beta\|_\beta^\beta}{\|f_\beta\|_1}\right)_+^{\frac{1}{\beta-1}} \tag{2.57}$$

where we have used $K_\beta^{\kappa,\lambda,\mu}(q)\equiv\iint\frac{f_{\beta;\,\kappa,\lambda,\mu}(p',q')}{|q-q'|}\,d^3p'\,d^3q'$. If we choose for κ and λ:

$$\kappa=\left(3-\frac{3}{\beta}\right)\|f_\beta\|_1^{2\beta-1}\|f_\beta\|_\beta^\beta\frac{\mathcal{E}_p^u(f_\beta)^{3\beta-3}}{(-\mathcal{E}_q(f_\beta))^{3\beta-2}}, \tag{2.58}$$

$$\lambda=\left(3-\frac{3}{\beta}\right)^{-1}\frac{\|f_\beta\|_1^{2-2\beta}}{\|f_\beta\|_\beta^\beta}\frac{(-\mathcal{E}_q(f_\beta))^{3\beta-3}}{\mathcal{E}_p^u(f_\beta)^{3\beta-4}}, \tag{2.59}$$

then we get the following:

$$f_{\beta;\,\kappa,\lambda,\mu}(p,q) \;=\; (\phi_{\beta;\,\kappa,\lambda,\mu}(q)-|p|)_+^{\frac{1}{\beta-1}}, \tag{2.60}$$

where

$$\phi_{\beta;\,\kappa,\lambda,\mu}(q)\equiv\iint\frac{f_{\beta;\,\kappa,\lambda,\mu}(p',q')}{|q-q'|}\,d^3p'\,d^3q'-\varkappa, \tag{2.61}$$

and where the constant $\varkappa$ can be determined from the constraint

$$\|f_{\beta;\,\kappa,\lambda,\mu}\|_1=1. \tag{2.62}$$

From this point forward, we drop the subscripts $\kappa,\lambda,$ and μ as we now focus our attention on this particular minimizer.

Next, we determine ϕ_β in more detail. We begin by computing the marginal mass distribution over configuration space:

$$\rho_\beta(q) \;\equiv\; \int f_\beta(p,q)\;d^3p, \tag{2.63}$$

$$=\; 4\pi\int_0^{\phi_\beta(q)_+}(\phi_\beta(q)-|p|)^{\frac{1}{\beta-1}}\;|p|^2d|p|, \tag{2.64}$$

$$=\; \frac{8\pi(\beta-1)^3}{\beta(2\beta-1)(3\beta-2)}(\phi_\beta(q))_+^{\frac{3\beta-2}{\beta-1}}, \tag{2.65}$$

where the last line follows by successive integration by parts. By definition,

$$\phi_\beta(q) \;\equiv\; \int\int\frac{f_\beta(p',q')}{|q-q'|}\;d^3p'\,d^3q'-\varkappa \tag{2.66}$$

$$=\; \frac{8\pi(\beta-1)^3}{\beta(2\beta-1)(3\beta-2)}\int\frac{(\phi_\beta(q'))_+^{\frac{3\beta-2}{\beta-1}}}{|q-q'|}\,d^3q'-\varkappa; \tag{2.67}$$

so upon taking the negative Laplacian of both sides we have

$$-\triangle\phi_\beta(q) \quad = \quad c(\beta)\,(\phi_\beta(q))_+^{\frac{3\beta-2}{\beta-1}} \tag{2.68}$$

with

$$c(\beta) \equiv \frac{32\pi^2(\beta-1)^3}{\beta(2\beta-1)(3\beta-2)}. \tag{2.69}$$

Our previous arguments with spherically symmetric equi-measurable rearrangements show that we can assume ϕ_β is spherically symmetric (where we take the center of the distribution to be the origin of our coordinate system). We write

$$\phi_\beta(q) = \phi_\beta(|q|) = \phi_\beta(r). \tag{2.70}$$

Let R_β be the first zero of $\phi_\beta(r)$. Then on the ball $|q| \le R_\beta$ the partial differential equation above becomes

$$\begin{aligned} \frac{d^2\phi_\beta}{dr^2} + \frac{2}{r}\frac{d\phi_\beta}{dr} + c(\beta)\,(\phi_\beta)_+^{\frac{3\beta-2}{\beta-1}} &= 0, \\ \frac{d\phi_\beta}{dr}(0) &= 0, \\ \frac{d\phi_\beta}{dr}(R_\beta) &= -\frac{1}{R_\beta^2}. \end{aligned} \tag{2.71}$$

Note that the first boundary condition is forced by the differential equation if we are to have a finite solution at the origin. The second boundary condition will ensure that f_β integrates to one (which also specifies the constant $\varkappa$ which we have conveniently absorbed into the definition of ϕ_β).

Noting that the equation (2.68) essentially gives the gravitational potential of our mass distribution (up to a sign difference), outside the ball $|q| \le R_\beta$ we must have

$$\phi_\beta(q) = \frac{1}{|q|} - \frac{1}{R_\beta}. \tag{2.72}$$

We note that as a potential function, ϕ_β is zero at the boundary of the mass distribution and not at infinity (as is usually the case). This fact is easy to forget and can be the source of many headaches!

Since for each $\beta > \frac{3}{2}$ the associated density f_β has compact support (c.f. section 5), we can conclude that $f_\beta \in \mathfrak{P}_1 \cap \mathfrak{L}^\beta$ as promised in the previous section. Hence, we conclude that $\mathcal{C}_\beta$ (the infimum over $f_\beta \in \mathfrak{P}_1 \cap \mathfrak{L}^\beta$) is equal to the infimum over the larger space Ω_β, and both are given by $\Phi_\beta(f_\beta)$. Note that this is also true for the critical case $\beta = \frac{3}{2}$ as shown in [KTZ08].

2.4 Calculation of $\mathcal{C}_\beta$

We begin by recalling the calculation above (c.f. (2.65)) for the spatial mass density associated to f_β (that is, the marginal distribution over q-space):

$$\rho_\beta(q) = \frac{8\pi(\beta-1)^3}{\beta(2\beta-1)(3\beta-2)} \left(\phi_\beta(q)\right)_+^{\frac{3\beta-2}{\beta-1}}.$$

2.4.1 A Most Useful Identity

We begin with a simple integration by parts on the defining PDE for ϕ_β:

$$c(\beta)\int_{B_{R_\beta}(0)} (\phi_\beta(q))^{\frac{4\beta-3}{\beta-1}}\ d^3q \ = \ \int_{B_{R_\beta}(0)} |\nabla\phi_\beta(q)|^2\ d^3q, \tag{2.73}$$

since $\phi_\beta(q) = 0$ when $|q| = R_\beta$.

We pair this with the Pohozaev identity:

$$\int_{B_{R_\beta}(0)} |\nabla\phi_\beta(q)|^2\ d^3q = \frac{6\beta-6}{4\beta-3} c(\beta)\int_{B_{R_\beta}(0)} (\phi_\beta(q))^{\frac{4\beta-3}{\beta-1}}\ d^3q - \frac{4\pi}{R_\beta}. \tag{2.74}$$

This identity can be seen by first noting

$$c(\beta)\int_{B_{R_\beta}(0)} (q\cdot\nabla\phi_\beta(q))\,(\phi_\beta(q))^{\frac{3\beta-2}{\beta-1}}\, d^3q = \int_{B_{R_\beta}(0)} (q\cdot\nabla\phi_\beta(q))\,(-\triangle\phi_\beta(q))\, d^3q. \tag{2.75}$$

The left-hand integral is fairly easy:

$$\int_{B_{R_\beta}(0)} (q\cdot\nabla\phi_\beta(q))\,(\phi_\beta(q))^{\frac{3\beta-2}{\beta-1}}\ d^3q = \frac{-3\beta+3}{4\beta-3}\int_{B_{R_\beta}(0)} (\phi_\beta(q))^{\frac{4\beta-3}{\beta-1}}\ d^3q. \tag{2.76}$$

The right-hand is more involved, and after a lengthy calculation yields:

$$\int_{B_{R_\beta}(0)} (q\cdot\nabla\phi_\beta(q))\,(-\triangle\phi_\beta(q))\ d^3q = -\frac{1}{2}\int_{B_{R_\beta}(0)} |\nabla\phi_\beta(q)|^2\ d^3q - \frac{2\pi}{R_\beta}. \tag{2.77}$$

Finally, combining these two expressions for the Dirichlet integral gives

$$\int_{B_{R_\beta}(0)} |\nabla\phi_\beta(q)|^2\ d^3q \ = \ \frac{4\pi}{R_\beta}\left(\frac{4\beta-3}{2\beta-3}\right), \tag{2.78}$$

or equivalently

$$c(\beta) \int (\phi_\beta(q))_+^{\frac{4\beta-3}{\beta-1}} \ d^3q \quad = \quad \frac{4\pi}{R_\beta} \left(\frac{4\beta-3}{2\beta-3} \right). \tag{2.79}$$

As we show below, this identity makes computation of the functionals comprising Φ_β remarkably easy!

2.4.2 The $\mathfrak{L}^\beta$ Norm

$$\begin{aligned}
\|f_\beta\|_\beta^\beta &= \iint (f_\beta(p,q))^\beta \ d^3p \, d^3q & (2.80)\\
&= 4\pi \iint_0^{\phi_\beta(q)_+} (\phi_\beta(q)_+ - |p|)^{\frac{\beta}{\beta-1}} \ |p|^2 \, d|p| \, d^3q & (2.81)\\
&= \frac{8\pi(\beta-1)}{2\beta-1} \iint_0^{\phi_\beta(q)_+} (\phi_\beta(q)_+ - |p|)^{\frac{2\beta-1}{\beta-1}} \ |p| \, d|p| \, d^3q & (2.82)\\
&= \frac{8\pi(\beta-1)^2}{(2\beta-1)(3\beta-2)} \iint_0^{\phi_\beta(q)_+} (\phi_\beta(q)_+ - |p|)^{\frac{3\beta-2}{\beta-1}} \ d|p| \, d^3q & (2.83)\\
&= \frac{8\pi(\beta-1)^3}{(2\beta-1)(3\beta-2)(4\beta-3)} \int (\phi_\beta(q)_+)^{\frac{4\beta-3}{\beta-1}} \ d^3q & (2.84)\\
&= \frac{1}{R_\beta} \left(\frac{\beta}{2\beta-3} \right). & (2.85)
\end{aligned}$$

2.4.3 The Ultrarelativistic Kinetic Energy

$$\begin{aligned}
\mathcal{E}_p^u(f_\beta) &= \iint |p| f_\beta(p,q) \ d^3p \, d^3q & (2.86)\\
&= 4\pi \iint_0^{\phi_\beta(q)_+} (\phi_\beta(q)_+ - |p|)^{\frac{1}{\beta-1}} \ |p|^3 \, d|p| \, d^3q & (2.87)\\
&= \frac{12\pi(\beta-1)}{\beta} \iint_0^{\phi_\beta(q)_+} (\phi_\beta(q)_+ - |p|)^{\frac{\beta}{\beta-1}} \ |p|^2 \, d|p| \, d^3q & (2.88)\\
&= \frac{24\pi(\beta-1)^2}{\beta(2\beta-1)} \iint_0^{\phi_\beta(q)_+} (\phi_\beta(q)_+ - |p|)^{\frac{2\beta-1}{\beta-1}} \ |p| \, d|p| \, d^3q & (2.89)\\
&= \frac{24\pi(\beta-1)^3}{\beta(2\beta-1)(3\beta-2)} \iint_0^{\phi_\beta(q)_+} (\phi_\beta(q)_+ - |p|)^{\frac{3\beta-2}{\beta-1}} \ d|p| \, d^3q & (2.90)\\
&= \frac{24\pi(\beta-1)^4}{\beta(2\beta-1)(3\beta-2)(4\beta-3)} \int (\phi_\beta(q)_+)^{\frac{4\beta-3}{\beta-1}} \ d^3q & (2.91)\\
&= \frac{3(\beta-1)}{4\pi(4\beta-3)} c(\beta) \int (\phi_\beta(q)_+)^{\frac{4\beta-3}{\beta-1}} \ d^3q & (2.92)\\
&= \frac{3\beta-3}{R_\beta(2\beta-3)}. & (2.93)
\end{aligned}$$

2.4.4 The Potential Energy

Our choice of scaling is meant to ensure that the potential energy of our minimizers equals their ultra-relativistic kinetic energy. To check this, we calculate the potential energy directly.

$$\begin{aligned}
-\mathcal{E}_q(f_\beta) &= \frac{1}{2}\iiiint \frac{f_\beta(p',q')f_\beta(p,q)}{|q-q'|}\, d^3p'\, d^3p\, d^3q'\, d^3q && (2.94)\\
&= \frac{1}{2}\iint \frac{\rho_\beta(q')\rho_\beta(q)}{|q-q'|}\, d^3q'\, d^3q && (2.95)\\
&= -\frac{(\beta-1)^3}{\beta(2\beta-1)(3\beta-2)}\iint \frac{(\phi_\beta(q)_+)^{\frac{3\beta-2}{\beta-1}}\,\triangle_{q'}\phi_\beta(q')}{|q-q'|}\, d^3q'\, d^3q && (2.96)\\
&= \frac{(\beta-1)^3}{\beta(2\beta-1)(3\beta-2)}\int_{B_{R_\beta}(0)}(\phi_\beta(q))^{\frac{3\beta-2}{\beta-1}}\left[\int_{B_{R_\beta}(0)}\frac{-\triangle_{q'}\phi_\beta(q')}{|q-q'|}d^3q'\right] d^3q. && (2.97)
\end{aligned}$$

We first work on the bracketed integral before tackling the entire expression:

$$\begin{aligned}
\int_{B_{R_\beta}(0)}\frac{-\triangle_{q'}\phi_\beta(q')}{|q-q'|}\, d^3q' &= \int_{B_{R_\beta}(0)}\nabla_{q'}\phi_\beta(q')\cdot\nabla_{q'}\left(\frac{1}{|q-q'|}\right) d^3q'\\
&\quad -\int_{\partial B_{R_\beta}(0)}\left(\frac{1}{|q-q'|}\right)\nabla_{q'}\phi_\beta(q')\cdot\, d\vec{\sigma'} && (2.98)\\
&= \int_{B_{R_\beta}(0)}\phi_\beta(q')\triangle_{q'}\left(\frac{-1}{|q-q'|}\right) d^3q'\\
&\quad +\int_{\partial B_{R_\beta}(0)}\phi_\beta(q')\,\nabla_{q'}\left(\frac{1}{|q-q'|}\right)\cdot\, d\vec{\sigma'}\\
&\quad -\int_{\partial B_{R_\beta}(0)}\left(\frac{1}{|q-q'|}\right)\nabla_{q'}\phi_\beta(q')\cdot\, d\vec{\sigma'} && (2.99)\\
&= 4\pi\left(\phi_\beta(q)+\frac{1}{R_\beta}\right). && (2.100)
\end{aligned}$$

Inserting this expression into the functional above gives:

$$\begin{aligned}
-\mathcal{E}_q(f_\beta) &= \frac{4\pi(\beta-1)^3}{\beta(2\beta-1)(3\beta-2)}\int_{B_{R_\beta}(0)}(\phi_\beta(q))^{\frac{3\beta-2}{\beta-1}}\left[\phi_\beta(q)+\frac{1}{R_\beta}\right] d^3q && (2.101)\\
&= \frac{c(\beta)}{8\pi}\int_{B_{R_\beta}(0)}(\phi_\beta(q))^{\frac{4\beta-3}{\beta-1}}\, d^3q+\frac{1}{2R_\beta}\int_{B_{R_\beta}(0)}\rho_\beta(q)\, d^3q && (2.102)\\
&= \frac{1}{2R_\beta}\left(\frac{4\beta-3}{2\beta-3}\right)+\frac{1}{2R_\beta} && (2.103)\\
&= \frac{3\beta-3}{R_\beta(2\beta-3)}. && (2.104)
\end{aligned}$$

2.4.5 The Formula for $\mathcal{C}_\beta$

We see from our work above that the ultra-relativistic kinetic energy and potential energy are indeed equal for f_β (as we chose in our scaling). Thus, the only contribution to $\mathcal{C}_\beta$ comes from the $\mathfrak{L}^\beta$ norm. Hence, we arrive at the following formula:

$$\mathcal{C}_\beta = \left(\frac{\beta}{R_\beta(2\beta - 3)}\right)^{\frac{1}{\beta}}. \tag{2.105}$$

2.5 $\mathcal{C}_\beta$ and the Standard Lane-Emden Polytropes

Though the formula for $\mathcal{C}_\beta$ given above is rather elegant, R_β is only defined implicity by our requirement that $\|f_\beta\|_1 = 1$. In order to compute $\mathcal{C}_\beta$ we rewrite its formula in terms of the solutions of the famous *Lane-Emden ODE* with standard initial data. In the usual notation (as in [Ch67]) this ODE is

$$\begin{aligned} \frac{d^2\theta_n}{d\xi^2} + \frac{2}{\xi}\frac{d\theta_n}{d\xi} + \theta_n^n &= 0, \\ \theta_n(0) &= 1, \\ \frac{d\theta_n}{d\xi}(0) &= 0. \end{aligned} \tag{2.106}$$

The solution to this ODE for a particular choice of n is often referred to as *the standard polytrope of index n*. It is well-known (c.f. [Ch67]) that the standard polytropes for $n \in [0, 5)$ first cross the ξ-axis at a finite distance from the origin. This first zero is often denoted ξ_n.

Explicit solutions are only known for three indices:

$$\theta_0(\xi) = 1 - \frac{\xi^2}{6}, \tag{2.107}$$

$$\theta_1(\xi) = \frac{\sin(\xi)}{\xi}, \tag{2.108}$$

$$\theta_5(\xi) = \frac{1}{\sqrt{1 + \frac{1}{3}\xi^2}}, \tag{2.109}$$

giving $\xi_0 = \sqrt{6}$, $\xi_1 = \pi$, and $\xi_5 = \infty$. Of equal importance is the slope of θ_n at the first zero. In the cases above, we have:

$$\frac{d\theta_0}{d\xi}(\xi_0) = -\frac{\sqrt{6}}{3}, \tag{2.110}$$

$$\frac{d\theta_1}{d\xi}(\xi_1) = -\frac{1}{\pi}, \tag{2.111}$$

$$\lim_{\xi\to\infty}\frac{d\theta_5}{d\xi}(\xi) = 0. \tag{2.112}$$

We next explore rescaling the standard polytropes in order to find functions which satisfy our equation for ϕ_β (2.71). We first note that the polytropic indices arising in the determination of $\mathcal{C}_\beta$ range over $(3,5]$ (so that $n = 1$ is clearly avoided in our considerations). We make the following definition (for $n \neq 1$)

$$\gamma_n(\xi) \equiv \alpha_n^{-1} A_n^{\frac{2}{n-1}} \theta_n(A_n\xi), \tag{2.113}$$

and note the following consequences:

$$\gamma_n\left(\frac{\xi_n}{A_n}\right) = 0, \tag{2.114}$$

$$\frac{d\gamma_n}{d\xi}(\xi) = \alpha_n^{-1} A_n^{\frac{n+1}{n-1}} \frac{d\theta_n}{d\xi}(A_n\xi), \tag{2.115}$$

$$\frac{d^2\gamma_n}{d\xi^2}(\xi) = \alpha_n^{-1} A_n^{\frac{2n}{n-1}} \frac{d^2\theta_n}{d\xi^2}(A_n\xi). \tag{2.116}$$

These change our ODE to

$$\alpha_n A_n^{\frac{-2n}{n-1}} \frac{d^2\gamma_n}{d\xi^2} + \frac{2}{A_n\xi}\alpha_n A_n^{-\frac{n+1}{n-1}} \frac{d\gamma_n}{d\xi} + \alpha_n^n A_n^{\frac{-2n}{n-1}} \gamma_n^n = 0, \tag{2.117}$$

which reduces to

$$\frac{d^2\gamma_n}{d\xi^2} + \frac{2}{\xi}\frac{d\gamma_n}{d\xi} + \alpha_n^{n-1}\gamma_n^n = 0, \tag{2.118}$$

$$\gamma_n(0) = \alpha_n^{-1} A_n^{\frac{2}{n-1}}, \tag{2.119}$$

$$\frac{d\gamma_n}{d\xi}(0) = 0. \tag{2.120}$$

We clearly must have

$$n(\beta) = \frac{3\beta - 2}{\beta - 1}, \tag{2.121}$$

where n runs from 5 down to 3 as β runs from $\frac{3}{2}$ up to infinity. Equivalently, we have

$$\beta(n) = \frac{n-2}{n-3}. \tag{2.122}$$

It is also clear that we will need

$$\alpha_{n(\beta)} = c(\beta)^{\frac{1}{n(\beta)-1}}. \tag{2.123}$$

The determination of $A_{n(\beta)}$ comes from the second boundary condition of (2.71):

$$\begin{aligned} \frac{d\gamma_{n(\beta)}}{d\xi}\left(\frac{\xi_{n(\beta)}}{A_{n(\beta)}}\right) &= \alpha_{n(\beta)}^{-1} A_{n(\beta)}^{\frac{n(\beta)+1}{n(\beta)-1}} \frac{d\theta_{n(\beta)}}{d\xi}(\xi_{n(\beta)}) & (2.124) \\ &= -\frac{A_{n(\beta)}^2}{\xi_{n(\beta)}^2}. & (2.125) \end{aligned}$$

Some algebra reveals that

$$A_{n(\beta)} = \left(\frac{-\xi_{n(\beta)}^2 \theta'_{n(\beta)}(\xi_{n(\beta)})}{\alpha_{n(\beta)}}\right)^{\beta}, \tag{2.126}$$

which leads to the following formula for R_β in terms of the standard Lane-Emden data:

$$R_\beta = \xi_{n(\beta)} \left(-\xi_{n(\beta)}^2 \theta'_{n(\beta)}(\xi_{n(\beta)})\right)^{1-2\beta} (c(\beta))^{\beta-1}. \tag{2.127}$$

This in turn leads to the following very useful (but decidedly more cumbersome) formula for $\mathcal{C}_\beta$:

$$\mathcal{C}_\beta = \left(\frac{\beta(-\xi_{n(\beta)}^2 \theta'_{n(\beta)}(\xi_{n(\beta)}))^{2\beta-1}}{(2\beta-3)\xi_{n(\beta)}(c(\beta))^{\beta-1}}\right)^{\frac{1}{\beta}}. \tag{2.128}$$

Incidentally, we have the following formula for ϕ_β :

$$\phi_\beta(q) = \alpha_{n(\beta)}^{-1} A_{n(\beta)}^{\frac{2}{n(\beta)-1}} \theta_{n(\beta)}(A_{n(\beta)} q),$$

which we do not expand further for reasons of brevity!

2.6 Numerical Results

It will be beneficial to rewrite our formula for $\mathcal{C}_\beta$ in terms of the standard polytropic index $n(\beta)$:

$$\mathcal{C}_\beta = \left(\frac{(n(\beta)-2)(-\xi_{n(\beta)}^2 \theta'_{n(\beta)}(\xi_{n(\beta)}))^{\frac{n(\beta)-1}{n(\beta)-3}}}{(5-n(\beta))\xi_{n(\beta)}(c(\beta))^{\frac{1}{n(\beta)-3}}}\right)^{\frac{n(\beta)-3}{n(\beta)-2}}. \tag{2.129}$$

We first compute the value of $\mathcal{C}_\beta$ as $\beta = 3/2$.

First, recall the following facts ([Bu78]):

$$\lim_{n\to 5} -\xi_n^2\theta_n'(\xi_n) = \sqrt{3}, \tag{2.130}$$

$$\lim_{n\to 5}(5-n)\xi_n = \frac{32\sqrt{3}}{\pi}. \tag{2.131}$$

Since $n(3/2) = 5$ and $c(3/2) = 8\pi^2/15$, we see that

$$\mathcal{C}_{\frac{3}{2}} = \frac{3}{8}\left(\frac{15}{16}\right)^{\frac{1}{3}}, \tag{2.132}$$

which reproduces the value found in [KTZ08] despite the fact that our analysis does not apply to the limiting case of non-compactly supported minimizers. Note that the polytrope of index 5 (commonly referred to as the Plummer Sphere in the astrophysical literature) is not compactly supported.

Using Maple to run the numerical approximations for the standard polytropes yields the following plot of $\mathcal{C}_\beta$ (displayed with the bounds found in [KTZ08] and a vertical line segment at $\beta = 3/2$ indicating that $\mathcal{C}_\beta = 0$ for $1 < \beta < 3/2$):

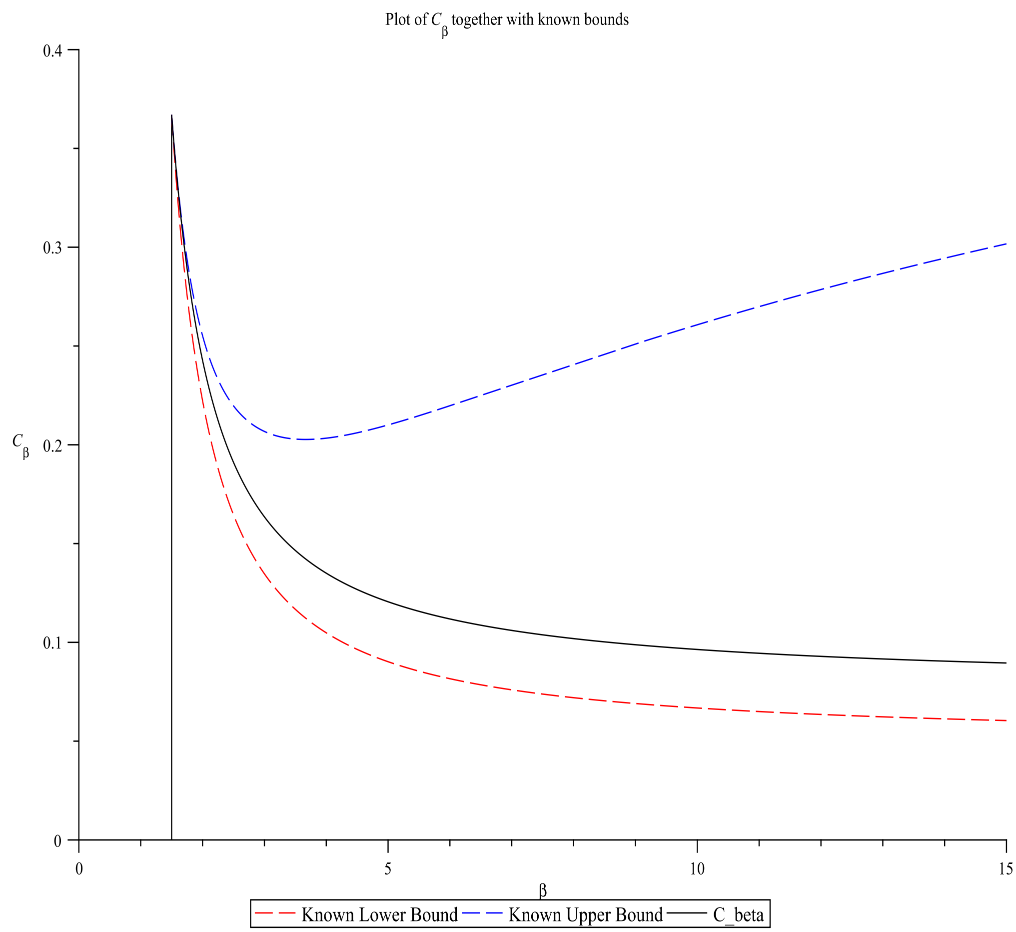

Figure 2.1: $\mathcal{C}_\beta$ together with known bounds

Incidentally, we can also give an improved upper bound over that listed in [KTZ08]. To see this, note that the upper bound in (2.5) is convex decreasing between $\beta = 3/2$ and $\beta \approx 3.6649$ beyond which it is strictly increasing for all β and converges to a finite value (namely $45/8\pi^2$) as β tends to infinity. In contrast, the numerical evaluation of $\mathcal{C}_\beta$ is a decreasing function of β. Hence we can improve the upper bound given in (2.5) by simply replacing it with its convex hull. We find (using Maple to estimate the minimum) that the improved upper bound takes the constant value 0.20269 for $\beta \geq 3.6649$.

2.7 Asymptotics

For the limiting behavior as β tends to infinity, we need that

$$\lim_{\beta\to\infty} c(\beta) = \frac{16\pi^2}{3}. \tag{2.133}$$

We also note that since $\lim_{\beta\to\infty} n(\beta) = 3$, the only terms that contribute are:

$$\mathcal{C}_\infty = \frac{3(-\xi_3^2\theta_3'(\xi_3))^2}{16\pi^2}. \tag{2.134}$$

This exact expression is a little less than illuminating since θ_3 is not known explicitly. However, there are extensive numerical data available. Referring to [Ho86, p. 407], we see that $-\xi_3^2\theta_3'(\xi_3) \approx 2.018236$. Thus, we can at least conclude that

$$\mathcal{C}_\infty \approx 0.077383. \tag{2.135}$$

In comparison, Theorem I of [GS85] (recalling that the total mass is 1 in our considerations) requires that initial data have $\mathcal{L}^\infty$-norm less than $40^{-3} \approx 0.00002$. Of course, Glassey and Schaeffer did not aim for the optimal constant and so were generous in their estimates. In comparison, the lower bound given in [KTZ08] is approximately 0.049438.

Since the standard polytrope of index 5 is known explicitly, we can find an asymptotic expression for $\mathcal{C}_\beta$ when β is sufficiently close to 3/2. We begin by examining an identity involving the zeroes of the standard polytrope of index n:

$$\frac{n+1}{(5-n)\xi_n} = \frac{\int_0^{\xi_n} (\theta_n(r))^{n+1}\, r^2\, dr}{(-\xi_n^2\theta_n'(\xi_n))^2}, \tag{2.136}$$

(this is essentially a reformulation of the identity found in 2.4.1 for the standard polytropes). The right-hand side limits to a finite value as n approaches 5 ([Bu78]):

$$\lim_{n\to 5} \frac{\int_0^{\xi_n} (\theta_n(r))^{n+1}\, r^2\, dr}{(-\xi_n^2\theta_n'(\xi_n))^2} = \frac{\pi\sqrt{3}}{16}. \tag{2.137}$$

So, for n sufficiently close to 5, we have that

$$\xi_n \approx \frac{16(n+1)}{\pi\sqrt{3}(5-n)}, \tag{2.138}$$

and accordingly, for β sufficiently close to $\frac{3}{2}$

$$R_\beta \approx \frac{16}{3\pi}\left(\frac{4\beta-3}{2\beta-3}\right)\left(\frac{32\pi^2(\beta-1)^3}{3\beta(2\beta-1)(3\beta-2)}\right)^{\beta-1}. \tag{2.139}$$

Finally, this yields an asymptotic expression for $\mathcal{C}_\beta$ near $\frac{3}{2}$:

$$\mathcal{C}_\beta \approx \left[\frac{3\pi}{16}\left(\frac{\beta}{4\beta-3}\right)\right]^{\frac{1}{\beta}}\left(\frac{3\beta(2\beta-1)(3\beta-2)}{32\pi^2(\beta-1)^3}\right)^{1-\frac{1}{\beta}}. \tag{2.140}$$

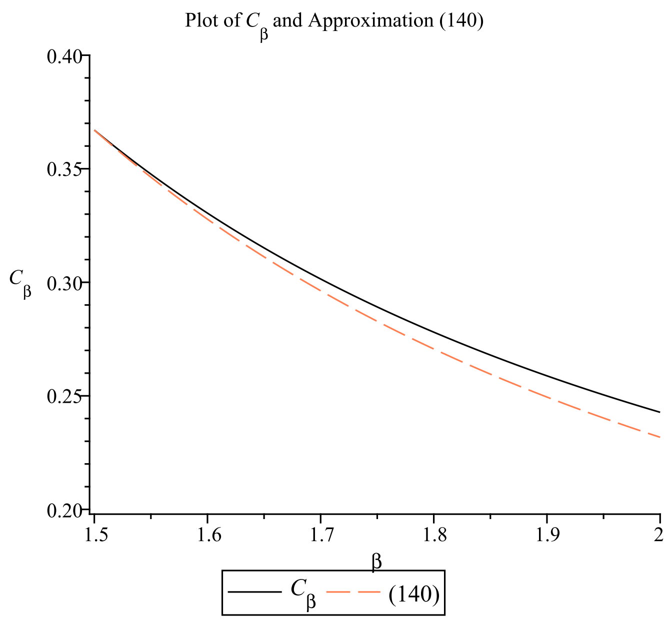

Figure 2.2: Approximation of $\mathcal{C}_\beta$ for β near 3/2

Chapter 3

Existence of Spherically Symmetric Initial Data with Zero Energy, Unit Mass, and Virial less than - 1/2 for the Relativistic Vlasov-Poisson Equation with Attractive Coupling

3.1 Introduction

As mentioned in the introductory chapter, much of the interest in the study of rVP^- is due to the fact that classical solutions can collapse in finite time. Thanks to the work of Pfaffelmoser [Pf92] on the global existence of classical solutions to the non-relativistic Vlasov-Poisson system (VP^-), we know that this collapse is due solely to presence of the relativistic velocity. In their seminal work on $\mathrm{rVP}^\pm$, Glassey and Schaeffer showed that global classical solutions to $\mathrm{rVP}^\pm$ will exist for initial data that are spherically symmetric, compactly supported in momentum space, and vanish on characteristics with vanishing angular momentum (hereafter abbreviated by I.D.) which are in addition compactly supported in $\mathbb{R}^6$ and have $\mathfrak{L}^\infty$-norm below a critical constant $\mathcal{C}_\infty^\pm$, with $\mathcal{C}_\infty^+ = \infty$ and $\mathcal{C}_\infty^- < \infty$. They also investigated what may happen when rVP$^-$ is launched by initial data with $\|f\|_\infty > \mathcal{C}_\infty^-$. They proved that negative energy data lead to "blow-up" (i.e. system collapse) in finite time. In [LMR08a], Lemou, Méhats, and Raphaël proved that collapse for systems launched by initial data with negative total energy approaches self-similar collapse. Around the same time, Kiessling and Tahvildar-Zadeh proved that any spherically symmetric classical solution of rVP$^-$ launched by initial data $f_0 \in \mathfrak{P}_3 \cap \mathfrak{C}^1$ † with *zero total energy* and total virial less than or equal to -1/2 will blow up in finite time (Theorem 6.1 of [KTZ08]).

Now, the authors of [KTZ08] did not give any specific examples, and recently the question was raised whether such data actually exist [Cal]. We here show that a plethora of such functions can be constructed from a not entirely obvious ansatz.

The plan of this brief note is as follows. To begin, we review the exact statement of Theorem 6.1 in [KTZ08] and give some indication of its proof. We then establish the basic formulae needed to carry out our subsequent investigations. We then show that the simplest ansatz (a nearly uniform ball of material centered at the origin) cannot have the desired properties. Next, we employ a core-halo structure (a central ball of material with a separate thin ring of material further out) to construct a class of initial data satisfying the requirements in [KTZ08]. In addition, we show that zero-energy initial data can have arbitrarily negative virial. Finally, we exhibit a second class of initial data with zero total energy and total virial less than or equal to -1/2 that have the nice property of being "singly supported."

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†By $\mathfrak{P}_3 \cap \mathfrak{C}^1$ we mean the set of probability measures on $\mathbb{R}^6$ whose first three moments are finite, that are absolutely continuous with respect to Lebesgue measure, and whose Radon-Nikodym derivative is $\mathfrak{C}^1$.

3.2 Blow-Up Criterion for Zero-Energy Initial Data

In this section, we outline the proof given in Theorem 6.1 [KTZ08] that initial data with zero total energy and total virial less than or equal to -1/2 will blow up in finite time.

Theorem 3.2.1 (Kiessling and Tahvildar-Zadeh 2008). *Let* $t \to f_t \in \mathfrak{P}_3 \cap \mathfrak{C}^1$ *be a spherically symmetric classical solution of* rVP^- *over some interval* $[0,T)$ *launched by initial data* f_0 *with*

$$\|f_0\|_{3/2} > \frac{3}{8}\left(\frac{15}{16}\right)^{1/3}.$$

If $\mathcal{E}(f_0) = 0$ *and* $\mathcal{V}(f_0) \leq -1/2$*, then* $T < \infty$*.*

Here we have used

$$\mathcal{E}(f_0) = \iint \sqrt{1+|p|^2} f(p,q) d^3p d^3q - \frac{1}{2} \iiiint \frac{f(p',q')f(p,q)}{|q-q'|} d^3p' d^3p d^3q' d^3q \quad (3.1)$$

$$\mathcal{V}(f_0) = \iint (q \cdot p) d^3p d^3q. \quad (3.2)$$

We present some basic facts from [KTZ08] that are essential to understanding the proof of the above theorem. First, the total energy and the $\mathfrak{L}^{3/2}$ norm are conserved by rVP$^{\pm}$ (as indeed are any other norm and a host of other functionals of the data - c.f. Proposition 2.1 of [KTZ08]). Thus, $\mathcal{E}(f_t) = 0$ and $\|f_t\|_{3/2} > \frac{3}{8}\left(\frac{15}{16}\right)^{1/3}$.

Next, Proposition 4.1 of [KTZ08] implies that any solution of rVP$^-$ launched by initial data f_0 with $\|f_0\|_{3/2} < \frac{3}{8}\left(\frac{15}{16}\right)^{1/3}$ must have strictly positive total energy. Hence, $\mathcal{E}(f_0) = 0$ already implies that $\|f_0\|_{3/2} \geq \frac{3}{8}\left(\frac{15}{16}\right)^{1/3}$. We will still need to check that it is strictly larger than this critical norm, but at least we know that the norm cannot possibly be smaller than this critical amount.

A final ingredient to the proof of the theorem above is the dynamical virial theorem:

$$\frac{d}{dt}\mathcal{V}(f_t) = \mathcal{E}(f_t) - \iint \frac{1}{\sqrt{1+|p|^2}} f_t(p,q) \, d^3p d^3q, \quad (3.3)$$

(presented in [KTZ08] as Proposition 2.3). In our particular case, $\mathcal{E}(f_t) = 0$, and so $\mathcal{V}(f_t)$ is a strictly decreasing function of t. Thus for any $t \in (\epsilon, T)$ we have

$$\mathcal{V}(f_t) < -C < -\frac{1}{2},$$

where C can depend on ϵ and f_0 (though this dependence is irrelevant for the purposes of the proof).

The blow-up result hinges upon two pieces of information from [GS85]:

1) For $t \to f_t \in \mathfrak{P}_3 \cap \mathfrak{C}^1$ which are classical solutions of rVP$^-$ over some time interval $t \in (0, T)$ we have

$$\frac{d}{dt}\iint |q|^2\sqrt{1+|p|^2}f_t(p,q)\; d^3pd^3q$$
$$= 2\mathcal{V}(f_t) - \iint |q|^2\frac{p}{\sqrt{1+|p|^2}}\cdot\nabla_q\varphi_t(q)f_t(p,q)\; d^3pd^3q,$$

2) Sphericity of the data implies

$$\left|\iint |q|^2\frac{p}{\sqrt{1+|p|^2}}\cdot\nabla_q\varphi_t(q)f_t(p,q)\; d^3pd^3q\right| \;\leq\; 1.$$

NOTE: Item 1) is identity (26) and item 2) is an estimate appearing in Theorem III in [GS85].

Integrating the equality in item 1) from 0 to T above and using the estimate in item 2) gives

$$\iint |q|^2\sqrt{1+|p|^2}f_T(p,q)\; d^3pd^3q$$
$$\leq \iint |q|^2\sqrt{1+|p|^2}f_0(p,q)\; d^3pd^3q + 2\int_0^T \mathcal{V}(f_t)dt + T$$
$$\leq \iint |q|^2\sqrt{1+|p|^2}f_0(p,q)\; d^3pd^3q + (2C-1)\epsilon - (2C-1)T.$$

Note that by assumption $2C > 1$. The l.h.s. above is positive no matter how large T becomes. The r.h.s. above is a fixed positive number minus a positive multiple of T. Thus, T cannot be too large or we get a contradiction. Thus, $T < \infty$.

3.3 Basic Formulae

We first establish the various formulae we shall need for our particular class of functions. Consider a generic separation-of-variables ansatz:

$$f(p,q) \;=\; \mathcal{C}\eta(|q|)\Phi(|p|)\mathscr{L}(\cos(\theta_{p,q})), \tag{3.4}$$

where $\theta_{p,q}$ is the angle between q and p (both considered as vectors in $\mathbb{R}^3$) and η, Φ, and $\mathscr{L}$ are non-negative. The coefficient $\mathcal{C}$ will be chosen to normalize the total mass to 1.

The total mass is given by

$$\begin{aligned}\iint f(p,q)\ d^3p\ d^3q =& \mathcal{C}\int_0^\infty \eta(|q|)|q|^2 d|q| \int_0^\infty \Phi(|p|)|p|^2 d|p| \iint_{\mathbb{S}^2\times\mathbb{S}^2} \mathscr{L}(\cos(\theta_{p,q}))d\Omega_p d\Omega_q \\ =& 8\pi^2\mathcal{C}\int_0^\infty \eta(|q|)|q|^2 d|q| \int_0^\infty \Phi(|p|)|p|^2 d|p| \int_{-1}^1 \mathscr{L}(x)dx.\end{aligned}$$

Using the notation

$$\|g\|_1 \equiv \int_0^\infty |g(r)|dr,$$

and recalling that all functions under consideration are non-negative we see that taking

$$\mathcal{C}^{-1} = 8\pi^2 \|\eta|q|^2\|_1 \|\Phi|p|^2\|_1 \int_{-1}^1 \mathscr{L}(x)dx$$

will ensure that f has total mass 1. Similarly, the $\mathfrak{L}^{3/2}$-norm is given by

$$\|f\|_{3/2} = \frac{\left(\int_0^\infty \eta(|q|)^{3/2}|q|^2 d|q| \int_0^\infty \Phi(|p|)^{3/2}|p|^2 d|p| \int_{-1}^1 \mathscr{L}(x)^{3/2}dx\right)^{2/3}}{2\pi^{2/3}\|\eta|q|^2\|_1 \|\Phi|p|^2\|_1 \int_{-1}^1 \mathscr{L}(x)dx}.$$

Recall that the energy and virial functionals are defined by

$$\begin{aligned}\mathcal{E}(f) \equiv& \iint \sqrt{1+|p|^2} f(p,q)\ d^3p\ d^3q \\ &- \frac{1}{2}\iiiint \frac{f(p',q')f(p,q)}{|q-q'|}\ d^3p'\ d^3p\ d^3q'\ d^3q \\ \mathcal{V}(f) \equiv& \iint q\cdot p\ f(p,q)\ d^3p\ d^3q.\end{aligned}$$

We begin by computing the virial:

$$\begin{aligned}\mathcal{V}(f) &= \iint q\cdot p f(p,q)\ d^3p\ d^3q \\ &= \mathcal{C}\int_0^\infty \eta(|q|)|q|^3 d|q| \int_0^\infty \Phi(|p|)|p|^3 d|p| \iint_{\mathbb{S}^2\times\mathbb{S}^2} \cos(\theta_{p,q})\mathscr{L}(\cos(\theta_{p,q}))d\Omega_p d\Omega_q \\ &= 8\pi^2\mathcal{C}\|\eta|q|^3\|_1\|\Phi|p|^3\|_1 \int_{-1}^1 x\mathscr{L}(x)dx \\ &= \frac{\|\eta|q|^3\|_1}{\|\eta|q|^2\|_1}\frac{\|\Phi|p|^3\|_1}{\|\Phi|p|^2\|_1}\frac{\int_{-1}^1 x\mathscr{L}(x)dx}{\int_{-1}^1 \mathscr{L}(x)dx}. \end{aligned} \tag{3.5}$$

Of course, in order to have a negative virial we must require

$$\int_{-1}^1 x\mathscr{L}(x)dx < 0.$$

For the energy, we compute the "kinetic" and potential energy contributions separately (the portion of the energy we label as "kinetic" also contains the rest mass energy). For the kinetic energy, we have

$$\begin{aligned} KE(f) &\equiv \iint \sqrt{1+|p|^2} f(p,q)\, d^3p\, d^3q \\ &= \mathcal{C} \int_0^\infty \Phi(|p|)\sqrt{1+|p|^2}|p|^2 d|p| \int_0^\infty \eta(|q|)|q|^2 d|q| \\ &\quad \cdot \iint_{\mathbb{S}^2\times\mathbb{S}^2} \mathscr{L}(\cos(\theta_{p,q}))d\Omega_p d\Omega_q \\ &= \frac{\|\Phi\sqrt{1+|p|^2}|p|^2\|_1}{\|\Phi|p|^2\|_1}. \end{aligned} \tag{3.6}$$

We begin computing the potential energy by first computing the spatial distribution ρ associated to f:

$$\begin{aligned} \rho(q) &\equiv \int f(p,q)\, d^3p \\ &= \mathcal{C}\eta(|q|) \int_0^\infty \Phi(|p|)|p|^2 d|p| \int_{\mathbb{S}^2} \mathscr{L}(\cos(\theta_{p,q}))d\Omega_p \\ &= \frac{\eta(|q|)}{4\pi\|\eta|q|^2\|_1} \end{aligned}$$

In terms of this function, the potential energy contribution is now given by

$$PE(f) \equiv -\frac{1}{2}\iint \frac{\rho(q)\rho(q')}{|q-q'|}\, d^3q'\, d^3q.$$

We calculate this integral directly, and after a little work find

$$PE(f) = -\frac{1}{\|\eta|q|^2\|_1^2} \int_0^\infty \eta(|q|)|q| \left(\int_0^{|q|} \eta(|q'|)|q'|^2 d|q'| \right) d|q|. \tag{3.7}$$

3.4 Characteristic Functions of Balls have Virial greater than $-1/2$

At this point, we can show that the simplest ansatz cannot work. Relaxing the differentiability requirement for a moment, we make the specific choices

$$\begin{aligned} \eta(|q|) &= \chi_{[0,R]}(|q|), \\ \Phi(|p|) &= \chi_{[0,P]}(|p|), \\ \mathscr{L}(x) &= \chi_{[-1,a]}(x) \end{aligned}$$

where $R > 0, P > 0, -1 < a < 1$, and χ_I is the characteristic function of the interval I. We have that

$$\begin{aligned} KE(f) &= \frac{3}{8}\left(\frac{\sqrt{1+P^2}}{P^2} + 2\sqrt{1+P^2} - \frac{\ln(P+\sqrt{1+P^2})}{P^3}\right), \\ PE(f) &= -\frac{3}{5R}. \end{aligned}$$

The zero-energy requirement allows us to solve for R in terms of P. Calculating the virial and plugging in the formula for $R = R(P)$ gives $\mathcal{V}(f)$ in terms of the parameters P and a. Simple asymptotics shows that for this ansatz

$$\mathcal{V}(f) > -\frac{9}{20}$$

for any choice of parameters. Smoothing out the boundaries of these step functions (taking care to keep the total energy zero) shows that this ansatz is untenable. Of course, this type of initial condition may well lead to collapse after finite time, but this cannot be verified by our virial condition.

3.5 A Core-Halo Ansatz

Having established formulae for the various quantities of interest and having ruled out the simplest possible ansatz we now proceed to try a core-halo scheme:

$$\eta(|q|) = \chi_{[0,R_1]}(|q|) + \alpha\chi_{[R_2,R_3]}(|q|), \tag{3.8}$$

$$\Phi(|p|) = \chi_{[0,P]}(|p|), \tag{3.9}$$

$$\mathscr{L}(x) = \chi_{[-1,a]}(x) \tag{3.10}$$

where $0 < R_1 \leq R_2 \leq R_3, 0 < \alpha, 0 < P$, and $-1 < a \leq 1$. Again, we have relaxed the differentiability requirement to make the computations tractable. The heuristic idea behind this ansatz is that the failure of (nearly) uniform balls to produce negative enough virial stems from needing most of the material concentrated near the origin - to ensure that the potential energy is sufficiently negative to balance the kinetic (and rest mass) energy of the system. Hence, the $q \cdot p$ term in the integral for the virial is small on average. Thus, putting a tiny halo of material far from the center should increase the magnitude of the virial while keeping the potential energy negative enough.

We first consider the specific choices

$$R_1 = \frac{1}{5}, R_2 = 1, R_3 = 2, \text{ and } P = 1.$$

The zero energy condition forces (thanks to Maple)

$$\alpha = \frac{1}{125}\frac{35\ln(1+\sqrt{2}) + 30 - 105\sqrt{2} + 2\sqrt{6480\sqrt{2} - 1655 - 2160\ln(1+\sqrt{2})}}{735\sqrt{2} - 188 - 245\ln(1+\sqrt{2})}$$

(a small but positive number). Again by Maple, choosing $-1 < a \leq -4/5$ gives a virial which is less than $-1/2$. Since our ansatz is compactly supported, we have the requisite number of moments. To complete the argument, we note that we can smooth out the various step functions in such a way that the resulting integrals are as close to the values obtained above as we like. Since α was chosen to force the zero-energy condition and the necessary value was strictly positive, it follows that by smoothing the test functions appropriately we can still choose an $\alpha > 0$ to keep the total energy zero. Using Maple to check the $\mathfrak{L}^{3/2}$-norm for the range of a listed above shows that our data are well above the critical norm $\frac{3}{8}\left(\frac{15}{16}\right)^{1/3}$. Thus, we have constructed an initial datum satisfying all the requirements of Theorem 6.1 in [KTZ08].

In addition, we give an argument showing that virial can be made arbitrarily negative. We make the following choices in (3.8):

$$\begin{aligned} R_1 &= P^{-2}, \\ R_2 &= P, \\ R_3 &= P^2. \end{aligned}$$

Now, the zero energy condition allows us to solve for $\alpha = \alpha(P)$. Using Maple, we see that as a function of P, α is positive for sufficiently large P and is proportional to $P^{-23/2}$ as $P \to \infty$. Plugging these choices into the formula for the virial and looking at the asymptotics for large P shows that the virial is proportional to $-(1-a)P^3$. Smoothing these functions out as above shows that the virial is unbounded below. As before, the $\mathfrak{L}^{3/2}$-norm for these functions remains well above the critical norm for a wide range of values for the parameter a.

3.6 A Monotonically Decreasing Ansatz

Having established the existence of initial data satisfying the theorem of Kiessling and Tahvildar-Zadeh, we are technically done. However, we can ask whether there are any monotonically decreasing family of initial data satisfying the same requirements. In other words, is the halo in the ansatz above really necessary? The answer is no. Even though nearly uniform balls (after regularization) was shown not to work, we can find zero-energy initial data with virial less than $-1/2$ that are monotonically decreasing and supported in a ball centered at the origin.

As in the previous considerations, we describe initial data that are not smooth. The understanding above that we can appropriately regularize our functions after the construction without significantly altering any of the required properties is still in force here. We now consider the ansatz

$$\eta(|q|) = \chi_{[0,R_1]}(|q|) + \left(\frac{R_1}{|q|}\right)^n \chi_{[R_1,R_2]}(|q|) + \left(\frac{R_1}{R_2}\right)^n \chi_{[R_2,R_3]}(|q|), \quad (3.11)$$

$$\Phi(|p|) = \chi_{[0,P]}(|p|), \quad (3.12)$$

$$\mathscr{L}(x) = \chi_{[-1,a]}(x). \quad (3.13)$$

Note that we have kept the same choices for the momentum and angular portions of f but substituted a spatial portion that is monotonically decreasing inside the ball of radius R_3 centered at the origin and zero outside this ball. A representative diagram for η is given in Figure 3.1 below. Note that it retains the essential ingredient that led us to the core-halo ansatz in the previous section. Namely, there is a dense ball of material centered about the origin (to raise the potential energy) surrounded by a thin shell of material further out (to increase the magnitude of the virial).

Even with this relatively simple ansatz, the calculations become rather messy. Thanks once again to Maple, we merely report a choice of parameters that does the job. Namely, choosing

$$R_1 = \frac{1}{100}, R_2 = \frac{1}{11}, R_3 = \frac{1}{10}, n = 3$$

sets a corresponding P (about 19.69) to force the zero-energy condition. Finally, any choice of a less than roughly $-9/10$ gives us a virial less than $-1/2$. A check of the

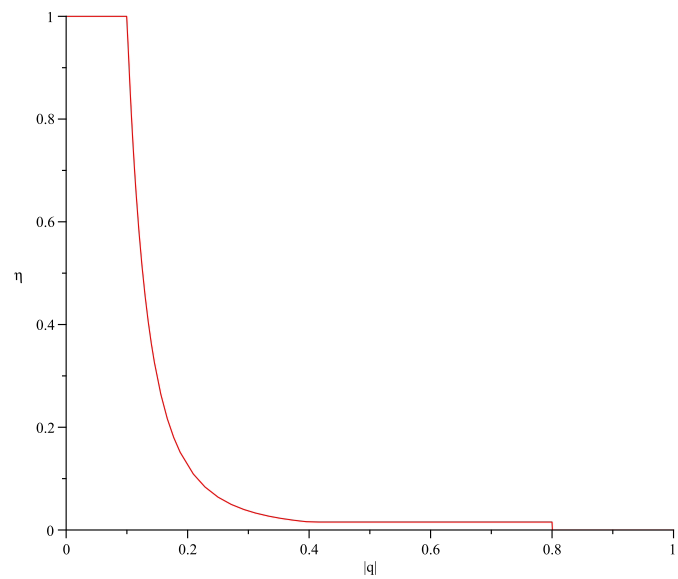


Figure 3.1: Representative monotonically decreasing η

$\mathfrak{L}^{3/2}$-norm shows that this solution is well above the critical value. Of course, the choice of parameters was totally arbitrary (motivated by guess-and-check), and so we expect an entire range of such parameters will do the job. However, we are content merely with the existence of one such solution!

Chapter 4

On the Relevance of the Relativistic Vlasov-Poisson System with Attractive Interactions for a Coulombic Two-Specie Plasma Model

4.1 Introduction

In this chapter, we will examine the proposal made in the appendix to [KTZ08] regarding a non-traditional approach to the derivation of the relativistic Vlasov-Poisson system with attractive coupling from a two-specie plasma model. In a nutshell, the authors envision an overall neutral two-specie plasma (where the charged species have equal mass but opposite charge) initially distributed identically, independently, and spherically symmetrically, and which interact via regularized Maxwell fields (to avoid the singularities plaguing electromagnetic self-interactions). As long as the distribution remains spherically symmetric, the magnetic contribution to the interaction should be negligible. This leaves the electrostatic interaction amongst the particles. Each particle then experiences an overall attractive force directed approximately toward the center of the distribution because each "+" charge sees the remaining system as having an overall extra "–" charge (and vice versa, of course) as the other charges roughly cancel. Hence, we could get rVP^- with perfectly central force field in the limit as the number of particles goes to infinity *on appropriate space-time scales*. Interestingly, this derivation *does not* depend on being in a weak-field regime!

In this chapter, we investigate this proposal using a simpler model in which particles interact only through (regularized) Coulombic interactions. Even though this type of model uses highly simplified physics for the underlying particle dynamics, its associated Vlasov model has been studied by plasma and astrophysicists in situations where magnetic effects are assumed to be negligible (see for instance [Sch94] for such an application). In our current work, we investigate this limited model as a first-step in the investigation of the fully electromagnetic model which we hope to address in future work. We note that in this case, the assumption of spherical symmetry is not required as the magnetic interactions are absent.

The remainder of the chapter is structured as follows. We start in §4.2 by introducing the regularized version of rVP^- that we will be working with. We will also recast the PDE as a fixed point problem and make some initial calculation of Lipschitz constants. We then give a brief review of the derivation obtained by Neunzert in §4.3. Though the original studies he undertook were for the non-relativistic Vlasov-Poisson system (appropriately regularized), the methods easily adapt to rVP^- (essentially we need only replace the classical formula for velocity in terms of momentum with the relativistic formula). Of course, the underlying dynamics is rather unphysical. In §4.4 we set up the two-specie plasma model that we will use throughout the remainder of the text. In a sense, this dynamics is also unphysical; yet, the Coulombic terms in our model will be present in a fully electromagnetic model (using the Coulomb gauge) while all other electromagnetic force terms are expected to be negligible with spherically symmetric initial data. In §4.5, we examine the rate of convergence of the initial data (as the number of particles N increases to infinity) through the study of Large Deviations. After this in §4.6, we introduce the different space-time scalings considered in the ensuing sections. In §4.7 we will study the infinitely many particles limit on the conventional space-time scale for Vlasov-Poisson dynamics. We shall see that the Vlasov limit exists on this scale, and yields the rVP system for an overall neutral two-specie plasma with "electrostatic" modes ([Sch94]). In this section we also comment on the important phenomenon of Landau Damping exhibited by some electrostatic modes. For the special choice of *iid* initial data jointly for all particles, we show that the traditional Vlasov dynamics reduces to that of freely streaming non-interacting particles. Finally in §4.8, we will discuss the typical behavior of the system in the infinitely many particles limit on the *a priori* N–body space-time scale. We will see that unlike the trivial free-streaming dynamics of the traditional two-specie relativistic Vlasov-Poisson system, a non-trivial limiting dynamics exists on the *a priori* scale featuring a rVP$^-$-type force term. However, we present evidence that some dissipative collision-type operator acts in conjunction with the rVP$^-$ operator in this $N \to \infty$ limit.

4.2 A Regularized Version of rVP$^-$

First, we fix a *spherically symmetric* function $\eta \in \mathscr{C}_c^\infty(\mathbb{R}^3)$ with $0 \leq \eta \leq 1$ and $\|\eta\|_1 = 1$. To be definite, we take

$$\text{supp}(\eta) \subseteq B_1(0).$$

For any $\epsilon > 0$ we define

$$\eta_\epsilon(q) = \frac{1}{\epsilon^3}\eta\left(\frac{q}{\epsilon}\right)$$

(note that $\|\eta_\epsilon\|_1 = 1 \ \ \forall \epsilon > 0$). We also define for convenience the *doubly regularized Coulomb force*:

$$G_\epsilon(q_1, q_2) \quad = \quad \iint \eta_\epsilon(q_1 - w)\frac{w - w'}{|w - w'|^3}\eta_\epsilon(w' - q_2)d^3w d^3w'. \tag{4.1}$$

Since $G_\epsilon(q_1, q_2) = G_\epsilon(q_1 - q_2, 0)$ (easily seen by a simple change of variables), we will typically use the slightly ambiguous notation

$$G_\epsilon(q_1 - q_2) \equiv G_\epsilon(q_1, q_2),$$

(since we can then write integrals involving G_ϵ as a convolution).

Let f_t be the solution to the following regularized version of rVP$-$

$$\partial_t f_t + v(p) \cdot \nabla_q f_t - G_\epsilon * \rho_{f_t}(q) \cdot \nabla_p f_t = 0, \tag{4.2}$$

where

$$\rho_{f_t}(q) = \int f_t(p,q) d^3 q \tag{4.3}$$

launched by some sufficiently regular f_0. Since we will assume the total mass of our system will be fixed at 1, we require $\iint f_0 = 1$ (which is then true for all later times). We denote this PDE by rVP$_\epsilon^-$. Thanks to the work of Neunzert (c.f. [Ne84] pp.60-74), we know that rVP$_\epsilon^-$ has a global, unique solution (at least in the weak sense). Moreover, the solution is classical so long as f_0 is continuously differentiable. This is in sharp contradistinction to the unregularized rVP$^-$ which is known to have finite time blow up for certain initial data (as described in previous chapters).

Associated to (4.2) is a time-dependent vector field on $\mathbb{R}^6$:

$$V^0[f.](t,p,q) \;=\; \begin{bmatrix} -G_\epsilon * \rho_{f_t}(q) \\ v(p) \end{bmatrix}. \tag{4.4}$$

$$\tag{4.5}$$

Associated to this vector-field is a flow on $\mathbb{R}^6$:

$$\begin{aligned} T^0_{t,0}[f.](p,q) &= (p(t), q(t)) \\ \begin{bmatrix} \dot p(t) \\ \dot q(t) \end{bmatrix} &= V^0[f.](t, p(t), q(t)) \\ (p(0), q(0)) &= (p,q). \end{aligned} \tag{4.6}$$

For reasons that will be apparent later, we refer to these objects as the *neutral vector field* and *neutral flow*, respectively. Note that $T^0_{t,0}[f.](p,q)$ traces out the characteristic curves for (4.2).

As with all Hamiltonian flows, (4.6) preserves phase space volume in $\mathbb{R}^6$. Having established the relevant definitions, we can now see that our PDE (4.2) is equivalent to the fixed point problem:

$$f_t(p,q) \;=\; f_0 \circ T^0_{0,t}[f.](p,q), \tag{4.7}$$

where $T^0_{0,t}$ is the backward flow on $\mathbb{R}^6$ (taking (p,q) to the initial condition $(\mathring{p},\mathring{q})$ where $T^0_{t,0}(\mathring{p},\mathring{q}) = (p,q)$). Simply put, the flow $T^0_{0,t}$ is associated to the vector field $-V^0$.

To conclude this section, we note some basic inequalities obeyed by $G_\epsilon, V^0[f.]$, and $T^0_{t,0}[f.]$. First, note the basic inequality

$$\left|\frac{p_1}{\sqrt{1+|p_1|^2}} - \frac{p_2}{\sqrt{1+|p_2|^2}}\right| \quad \leq \quad |p_1 - p_2|. \tag{4.8}$$

Also, $\eta \in \mathscr{C}^\infty_c(\mathbb{R}^3)$ has some Lipschitz constant L_η. It follows easily that η_ϵ has Lipschitz constant $\epsilon^{-4}L_\eta$.

We will need a sharp form of the Hardy-Littlewood-Sobolev Inequality ([LL01] p. 106): Let $f, g \in L^{3/2}(\mathbb{R}^3)$, then

$$\left|\iint f(x)|x-y|^{-2}g(y)d^3x d^3y\right| \leq (2\pi^2)^{\frac{2}{3}}\|f\|_{\frac{3}{2}}\|g\|_{\frac{3}{2}}. \tag{4.9}$$

Next,

$$\begin{aligned}
&\left|G_\epsilon(q_1-q') - G_\epsilon(q_2-q')\right| \\
&= \left|\iint \left(\eta_\epsilon(q_1{-}w) - \eta_\epsilon(q_2{-}w)\right)\frac{w-w'}{|w-w'|^3}\eta_\epsilon(w'{-}q')d^3w d^3w'\right| \\
&\leq \iint \frac{|\eta_\epsilon(q_1{-}w) - \eta_\epsilon(q_2{-}w)|\,\eta_\epsilon(w'{-}q')}{|w-w'|^2}d^3w d^3w' \\
&\leq (2\pi^2)^{\frac{2}{3}}\left(\int |\eta_\epsilon(q_1{-}w) - \eta_\epsilon(q_2{-}w)|^{\frac{3}{2}}\, d^3w\right)^{\frac{2}{3}}\|\eta_\epsilon\|_{\frac{3}{2}}
\end{aligned}$$

Since

$$\|\eta_\epsilon\|_{\frac{3}{2}} = \epsilon^{-1}\|\eta\|_{\frac{3}{2}} \leq \epsilon^{-1}\left(\frac{4\pi}{3}\right)^{\frac{2}{3}},$$

we have

$$\begin{aligned}
&\left|G_\epsilon(q_1-q') - G_\epsilon(q_2-q')\right| \\
&\leq \epsilon^{-2}(2\pi^2)^{\frac{2}{3}}\left(\frac{4\pi}{3}\right)^{\frac{2}{3}}\left(\int_{B_1(q_1/\epsilon)\cup B_1(q_2/\epsilon)}\left|\eta\left(\frac{q_1}{\epsilon}{-}w\right) - \eta\left(\frac{q_2}{\epsilon}{-}w\right)\right|^{\frac{3}{2}} d^3w\right)^{\frac{2}{3}} \\
&\leq \frac{1}{\epsilon^3}\left(\frac{8\pi^2}{3}\right)^{\frac{4}{3}} L_\eta|q_1-q_2|.
\end{aligned}$$

Hence, G_ϵ is Lipschitz continuous (separately in either of its arguments by symmetry) with Lipschitz constant

$$L_G = \frac{1}{\epsilon^3}\left(\frac{8\pi^2}{3}\right)^{\frac{4}{3}} L_\eta. \tag{4.10}$$

We can also use Hardy-Littlewood-Sobolev to show that G_ϵ is bounded:

$$\begin{aligned} |G_\epsilon(q_1-q_2)| &\leq \iint \eta_\epsilon(q_1-w)|w-w'|^{-2}\eta_\epsilon(w'-q_2)d^3wd^3w' \\ &\leq (2\pi^2)^{2/3}\|\eta_\epsilon\|^2_{\frac{3}{2}} = \frac{(2\pi^2)^{2/3}}{\epsilon^2}\|\eta\|^2_{\frac{3}{2}}. \end{aligned}$$

From our assumption on the regularizer η, we have

$$|G_\epsilon(q_1-q_2)| \leq \frac{1}{\epsilon^2}\left(\frac{32\pi^4}{9}\right)^{2/3}. \tag{4.11}$$

Note that the ratio of L_G to this bound on G_ϵ is of order ϵ^{-1}. Hence, we can always assume ϵ is taken small enough that L_G serves as a bound on both the Lipschitz constant and size of G_ϵ.

This nets us a t-independent Lipschitz constant for $V^0[f.]$:

$$\left|V^0[f.](t,p_1,q_1)-V^0[f.](t,p_2,q_2)\right| \leq L_0|(p_1,q_1)-(p_2,q_2)|, \tag{4.12}$$

where

$$L_0 = \max\{1, L_G\}. \tag{4.13}$$

By a standard application of Gronwall's Inequality (c.f. [Ro95] p.142), we have a Lipschitz constant for $T^0_{t,0}[f.]$:

$$\left|T^0_{t,0}[f.](p_1,q_1) - T^0_{t,0}[f.](p_2,q_2)\right| \leq e^{L_0 t}|(p_1,q_1)-(p_2,q_2)|. \tag{4.14}$$

We also note that both V^0 and $T^0_{t,0}$ are continuous as functions of t.

4.3 A Review of Neunzert's Technique

We first recall Neunzert's original derivation of the (regularized) Vlasov-Poisson system. We follow here the notes given in [Ne84] pp. 60 - 71, but modify the proof to include the relativistic velocity in terms of momentum. In this way, we actually get a derivation of rVP$^-_\epsilon$. As mentioned above (and as will become apparent shortly), the underlying physics describing the point particles is highly questionable.

We imagine a collection of N identical point particles (each having unit mass) interacting via Newtonian gravity. The motion of these particles is governed by the dynamics:

$$\begin{aligned} \dot{q_k}(t) &= v(p_k(t)) & (4.15) \\ \dot{p_k}(t) &= \sum_{i \neq k} \frac{q_i(t) - q_k(t)}{|q_i(t) - q_k(t)|^3}, & (4.16) \end{aligned}$$

where

$$v(p) = \frac{p}{\sqrt{1 + |p|^2}}. \tag{4.17}$$

Of course, the singularity present in the force terms prevents us from having global solutions. In fact, the lifetime of any particular solution may well decrease with the number of particles. So, to effect the derivation, we will need to regularize the gravitational force. Though there are a number of options for accomplishing this, the easiest method is by convolution with a sufficiently regular function. We doubly regularize the gravitational force by replacing the dynamics above with

$$\begin{aligned} \dot{q_k}(t) &= v(p_k(t)) & (4.18) \\ \dot{p_k}(t) &= \sum_{i=1}^{N} \iint \eta_\epsilon(q_i(t) - q) \frac{q - q'}{|q - q'|^3} \eta_\epsilon(q' - q_k(t)) d^3q d^3q'. & (4.19) \end{aligned}$$

Note that the sum now extends over all i (including $i = k$) since the regularization automatically forces the self-interaction term to be zero.

Since we are interested in an infinite particle limit, we have the annoying problem that our underlying phase space ($\mathbb{R}^{6N}$ in this case) changes with the number of particles (making the idea of limit problematic). The way to overcome this issue is to recast our N-particle system as an *empirical density* on $\mathbb{R}^6$ via

$$\Delta_t^{(N)}(p, q) \equiv \frac{1}{N} \sum_{i=1}^{N} \delta\left(p - p_i(t)\right) \delta\left(q - q_i(t)\right). \tag{4.20}$$

We denote the q-marginal of $\Delta_t^{(N)}$ (called the empirical spatial density) as

$$\rho_t^{(N)}(q) \equiv \frac{1}{N} \sum_{i=1}^{N} \delta\left(q - q_i(t)\right). \tag{4.21}$$

Note that both of these object are formally probability measures (on $\mathbb{R}^6$ and $\mathbb{R}^3$, respectively), but there is absolutely *no stochasticity in out model at this stage.* In standard fashion, we can recast our dynamics above as a PDE in terms of the measure:

$$\partial_t \Delta_t^{(N)} + v(p) \cdot \nabla_q \Delta_t^{(N)} - N G_\epsilon * \rho_t^{(N)} \cdot \nabla_p \Delta_t^{(N)} = 0. \tag{4.22}$$

The meaning of this equation is understood at the level of distributions. That is, begin by considering the time derivative of the action of $\Delta_t^{(N)}$ on some test function ϕ:

$$\partial_t \iint \phi(p,q)\Delta_t^{(N)}(p,q)d^3pd^3q = \frac{1}{N}\sum_{i=1}^{N} \partial_t\phi(p_i(t),q_i(t)).$$

Using the particle dynamics given by (4.18) and (4.19), integrating by parts, and rearranging the resulting terms gives (4.22).

A careful glance at (4.22) shows that we cannot hope for much of a limit due to the force term $NG_\epsilon * \rho_t^{(N)}$ which grows on the order of N as the number of particles increases. There are a number of ways to remedy this situation. In the non-relativistic setting, one can assume that the particles have mass $1/N$ (which is how Neunzert presents the idea). This is a little dubious from a physical point of view; after all, the physical particles being modeled have some definite mass - small perhaps but certainly non-zero. It is the "small" in the previous sentence that gives us a clue how to make this artifice more sound. Namely, our particles have fixed mass in some *a priori* system of units, but we change to a "macroscopic" unit of mass where our entire collection of particles has total mass one.

This is all well and good in a non-relativistic setting, but allowing the mass of the particles to tend to zero in our relativistic setting is very problematic (as the mass is incorporated into the relativistic velocity formula in a non-trivial way). We simply note here that removing the troublesome factor of N can also be effected in this setting by appropriate space-time rescalings (more details are presented in §4.6). On this appropriate "macroscopic" scale, we have the PDE

$$\partial_t\Delta_t^{(N)} + v(p)\cdot\nabla_q\Delta_t^{(N)} - G_\epsilon * \rho_t^{(N)}\cdot\nabla_p\Delta_t^{(N)} = 0, \tag{4.23}$$

where now the force term $G_\epsilon * \rho_t^{(N)}$ remains of order 1 independent of the number of particles (which is referred to as the so-called *mean-field approximation*). The similarity in form between (4.23) and (4.2) leads us to suspect that if $\Delta_0^{(N)}$ limits (in some suitable topology) to f_0, then the same should be true at later times.

The appropriate topology is given by the dual, bounded Lipschitz distance (denoted d_{bL^*}) on the space of probability measures: for two probability measures μ and ν defined on some space Ω

$$d_{bL^*}(\mu,\nu) = \sup_{\varphi\in\mathcal{D}(\Omega)} \left| \int \varphi d(\mu-\nu) \right|, \tag{4.24}$$

where

$$\mathcal{D}(\Omega) = \{\varphi : \Omega \to [-1,1] \mid \text{Lip}(\varphi) \leq 1\}. \tag{4.25}$$

See [Ne84] pp. 63-64 and the appendix to [EKR09] for more details.

This distance has a nice physical interpretation (for this analogy, it helps to think of those φ that are non-negative and smooth). Think of the test functions φ as being a collection of measurement devices which sample μ and ν (thought of here as mass distributions on Ω). The crucial aspect of the norm defined above is that all of the φ are constrained to have Lipschitz constant less than 1. Effectively, this means our test functions cannot be arbitrarily peaked about any particular point in Ω (as $\text{Lip}(\varphi) \leq 1$ controls the size of the derivative). Thus, the Lipschitz condition effectively creates a threshold in the space Ω below which differences in μ and ν are effectively smoothed over. This is precisely the sort of measurement we need! If we are allowed to zoom in on any scale, then there is no way a finite number of particles can ever be close to a continuum distribution of matter.

We should remark here that the topology generated by the dual, bounded Lipschitz distance is equivalent to the topology generated by all bounded, continuous functions. Namely, a sequence of probability measures $\{\nu_n\}$ converges to some probability measure ν iff for every bounded, continuous g

$$\int g d\nu_n \to \int g d\nu$$

(which follows easily as every bounded continuous function can be approximated by a bounded, Lipschitz continuous function). This topology goes by several different names (depending on the branch of mathematics you are consulting). Probabilists tend to call this the weak-$*$ topology, but analysts hesitate to use this terms since the space of probability measures is not dual to the space of bounded, continuous functions. The other name in use is *convergence in law*. Should we have need to reference this topology (other than through the metric d_{bL^*}) we shall adopt this name.

Suppose now that our initial condition is chosen so that

$$d_{bL^*}\left(\Delta_0^{(N)}, f_0\right) \to 0 \text{ as } N \to \infty.$$

We wish to prove that the same limit holds at any later time:

$$d_{bL^*}\left(\Delta_t^{(N)}, f_t\right) \to 0 \text{ as } N \to \infty.$$

This proof is accomplished in two basic steps. The first involves recasting the PDEs (4.23) and (4.2) as a fixed point problem (which we have accomplished for rVP$_\epsilon^-$ in (4.6)). The second is to use this characterization (and Gronwall's Inequality) to show that the dual, bounded Lipschitz distance at time t is bounded above by a constant times the distance at time 0 (with the constant depending on t). This shows the continuum limit of our particle dynamics is precisely the regularized version of rVP$^-$ given by (4.2).

Since (4.23) is formally the same as (4.2), its fixed point form is nothing more than

$$\Delta_t^{(N)}(p,q) \quad = \quad \Delta_0^{(N)} \circ T_{0,t}^0[\Delta_\cdot^{(N)}](p,q), \tag{4.26}$$

where the flow T^0 and its generating vector field V^0 are given just as in (4.6) and (4.4) (and of course replacing f_t by $\Delta_t^{(N)}$).

We now examine the convergence of $d_{bL^*}\left(\Delta_t^{(N)}, f_t\right)$. Let φ be an arbitrary element of $\mathcal{D}(\Omega)$. Then

$$\begin{aligned}
&\left|\iint \varphi(p,q)\left(\Delta_t^{(N)}(p,q) - f_t(p,q)\right) d^3p d^3q\right| \\
&\quad = \left|\iint \varphi(p,q)\left(\Delta_0^{(N)} \circ T_{0,t}^0[\Delta_\cdot^{(N)}](p,q) - f_0 \circ T_{0,t}^0[f_\cdot](p,q)\right) d^3p d^3q\right| \\
&\quad \le \left|\iint \varphi(p,q)\left(\Delta_0^{(N)} \circ T_{0,t}^0[\Delta_\cdot^{(N)}](p,q) - \Delta_0^{(N)} \circ T_{0,t}^0[f_\cdot](p,q)\right) d^3p d^3q\right| \\
&\qquad\quad + \left|\iint \varphi(p,q)\left(\Delta_0^{(N)} \circ T_{0,t}^0[f_\cdot](p,q) - f_0 \circ T_{0,t}^0[f_\cdot](p,q)\right) d^3p d^3q\right| \\
&\quad \le \iint \left|\varphi \circ T_{t,0}^0[\Delta_\cdot^{(N)}](p,q) - \varphi \circ T_{t,0}^0[f_\cdot](p,q)\right| \Delta_0^{(N)}(p,q) d^3p d^3q \\
&\qquad\quad + \left|\iint \varphi \circ T_{t,0}^0[f_\cdot](p,q)\left(\Delta_0^{(N)}(p,q) - f_0(p,q)\right) d^3p d^3q\right| \\
&\quad \le \iint \left|T_{t,0}^0[\Delta_\cdot^{(N)}](p,q) - T_{t,0}^0[f_\cdot](p,q)\right| \Delta_0^{(N)}(p,q) d^3p d^3q \\
&\qquad\quad + e^{L_0 t}\left|\iint e^{-L_0 t}\varphi \circ T_{t,0}^0[f_\cdot](p,q)\left(\Delta_0^{(N)}(p,q) - f_0(p,q)\right) d^3p d^3q\right|,
\end{aligned}$$

where in the last inequality we have used the defining property of $\mathcal{D}(\Omega)$ in the first integral, and the L_0 appearing in the second integral is the Lipschitz constant for the neutral vector field given by (4.13).

The second integral in the last inequality is bounded above by $d_{bL^*}\left(\Delta_0^{(N)}, f_0\right)$ since $\varphi \circ T^0_{t,0}[f_\cdot]$ has Lipschitz constant at most $\exp(L_0 t)$. Using this upper bound for the second integral and then taking the supremum over all φ in $\mathcal{D}(\Omega)$ gives

$$d_{bL^*}\left(\Delta_t^{(N)}, f_t\right) \quad \leq \quad e^{L_0 t} d_{bL^*}\left(\Delta_0^{(N)}, f_0\right) + \lambda(t), \tag{4.27}$$

where

$$\lambda(t) \quad \equiv \quad \iint \left| T^0_{t,0}[\Delta_\cdot^{(N)}](p,q) - T^0_{t,0}[f_\cdot](p,q)\right| \Delta_0^{(N)}(p,q) d^3p d^3q. \tag{4.28}$$

To make progress, we iterate the flow generating $T^0_{t,0}$:

$$\begin{aligned}
\lambda(t) \quad = \quad & \int_0^t \iint \Big| V^0[\Delta_\cdot^{(N)}](\tau, T^0_{\tau,0}[\Delta_\cdot^{(N)}](p,q)) \\
& \qquad -V^0[f_\cdot](\tau, T^0_{\tau,0}[f_\cdot](p,q))\Big| \Delta_0^{(N)}(p,q) d^3p d^3q d\tau \\
\leq \quad & \int_0^t \iint \Big| V^0[\Delta_\cdot^{(N)}](\tau, T^0_{\tau,0}[\Delta_\cdot^{(N)}](p,q)) \\
& \qquad -V^0[f_\cdot](\tau, T^0_{\tau,0}[\Delta_\cdot^{(N)}](p,q))\Big| \Delta_0^{(N)}(p,q) d^3p d^3q d\tau \\
& + \int_0^t \iint \Big| V^0[f_\cdot](\tau, T^0_{\tau,0}[\Delta_\cdot^{(N)}](p,q)) \\
& \qquad -V^0[f_\cdot](\tau, T^0_{\tau,0}[f_\cdot](p,q))\Big| \Delta_0^{(N)}(p,q) d^3p d^3q d\tau \\
\leq \quad & \int_0^t \iint \Big| V^0[\Delta_\cdot^{(N)}](\tau,p,q) - V^0[f_\cdot](\tau,p,q)\Big| \Delta_\tau^{(N)}(p,q) d^3p d^3q d\tau \\
& + L_0 \int_0^t \iint \Big| T^0_{\tau,0}[\Delta_\cdot^{(N)}](p,q) - T^0_{\tau,0}[f_\cdot](p,q)\Big| \Delta_0^{(N)}(p,q) d^3p d^3q d\tau \\
\leq \quad & \int_0^t \iint \Big| G_\epsilon * \left[\rho_{f_\tau} - \rho_\tau^{(N)}\right](q)\Big| \Delta_\tau^{(N)}(p,q) d^3p d^3q d\tau + L_0 \int_0^t \lambda(\tau) d\tau.
\end{aligned}$$

As we noted in the previous section, G_ϵ is both bounded above by and Lipschitz continuous with constant $L_G \leq L_0$. Since G_ϵ is a 3-dimensional vector of bounded, Lipschitz continuous functions, we have

$$\iint \left| G_\epsilon * \left[\rho_{f_\tau} - \rho_\tau^{(N)}\right](q)\right| \Delta_\tau^{(N)}(p,q) d^3p d^3q \leq \sqrt{3} L_0 d_{bL^*}\left(\Delta_\tau^{(N)}, f_\tau\right).$$

This leaves us with the inequality:

$$\lambda(t) \quad \leq \quad L_0 \int_0^t \lambda(\tau) d\tau + \sqrt{3} L_0 \int_0^t d_{bL^*}\left(\Delta_\tau^{(N)}, f_\tau\right) d\tau. \tag{4.29}$$

We now appeal to an obvious corollary to Gronwall's Inequality (c.f. [Ro95] pg. 142) which we note below for future use:

Theorem 4.3.1 (Gronwall's Inequality)**.** *Suppose that for non-negative real numbers α, β the integrable functions u, v satisfy the inequality*

$$u(t) \leq \alpha \int_0^t u(\tau)d\tau + \beta \int_0^t v(\tau)d\tau,$$

then u, v also satisfy the inequality

$$u(t) \quad \leq \quad \beta e^{\alpha t} \int_0^t v(\tau)e^{-\alpha\tau}d\tau.$$

Thus, (4.29) becomes

$$\lambda(t) \quad \leq \quad \sqrt{3}L_0 e^{L_0 t} \int_0^t d_{bL^*}\left(\Delta_\tau^{(N)}, f_\tau\right) e^{-L_0\tau}d\tau, \tag{4.30}$$

which gives the following inequality when combined with (4.27):

$$e^{-L_0 t}d_{bL^*}\left(\Delta_t^{(N)},f_t\right) \leq \sqrt{3}L_0 \int_0^t d_{bL^*}\left(\Delta_\tau^{(N)},f_\tau\right) e^{-L_0\tau}d\tau + d_{bL^*}\left(\Delta_0^{(N)},f_0\right). \tag{4.31}$$

Another application of Gronwall's Inequality yields:

$$d_{bL^*}\left(\Delta_t^{(N)},f_t\right) \quad \leq \quad \frac{e^{(\sqrt{3}+1)L_0 t}}{\sqrt{3}L_0} d_{bL^*}\left(\Delta_0^{(N)},f_0\right). \tag{4.32}$$

Thus, we have the following theorem:

Theorem 4.3.2 (essentially Neunzert 1975)**.** *Suppose that f_t is a classical solution of rVP_ϵ^-. Suppose also that initial data for the dynamics* (4.18) *and* (4.19) *is chosen so that*

$$\lim_{N\to\infty} d_{bL^*}\left(\Delta_0^{(N)}, f_0\right) = 0.$$

Then for all later times, we have

$$\lim_{N\to\infty} d_{bL^*}\left(\Delta_t^{(N)}, f_t\right) = 0.$$

4.4 A "Relativistic" N–Body Coulomb System

We now wish to apply the same techniques developed above to a two-specie plasma model with regularized electromagnetic interactions. The hope is that we get a similar result to Theorem 4.3.2 but for a slightly more relevant underlying dynamics. As stated in the introduction, we here only examine the situation regularized Coulombic interactions. We will see that simpler situation still has many hurdles to overcome in comparison to Neunzert's original proof.

4.4.1 The N–Body Dynamics

We consider a two-specie plasma with N particles of each type (positive and negative). For simplicity, we assume the particles have unit mass, unit magnitude charge, and label the particles so that even labels refer to the positively charged species, and hence the odd labels are negative charges. Define the charge indicator by

$$e_i = \begin{cases} +1 & i \text{ even} \\ -1 & i \text{ odd} \end{cases} .$$

The state of our plasma at any given time is described by the point

$$\mathcal{X}(t) \equiv (q_1(t), p_1(t), \ldots, q_{2N}(t), p_{2N}(t)) \in \mathbb{R}^{6N}$$

and is governed by the following coupled system of ODEs:

$$\dot{q}_i(t) = v(p_i(t)), \tag{4.33}$$

$$\dot{p}_i(t) = e_i \eta_\epsilon * E_t^\epsilon(q_i(t)), \tag{4.34}$$

where

$$v(p) = \frac{p}{\sqrt{1+|p|^2}}, \tag{4.35}$$

and

$$E_t^\epsilon(q) = \sum_{i=1}^{N} \int \left[\eta_\epsilon(q' - q_{2i}(t)) - \eta_\epsilon(q' - q_{2i-1}(t))\right] \frac{q-q'}{|q-q'|^3} d^3q', \tag{4.36}$$

where η_ϵ was defined in §4.2. We call the model a "relativistic" Coulomb system motivated solely by the use of the relativistic velocity in (4.33). Yet, the model is certainly not Lorentz covariant! Future work will be geared toward incorporating the magnetic interactions which will better justify our use of the term "relativistic" (though the regularization still breaks the symmetry).

Note that we have doubly regularized the charges: the charges are smeared out by η_ϵ both in the field (4.36) and in the force felt by each particle (4.34). The spherical symmetry of the regularizer η_ϵ ensures that the self-interaction terms are zero.

Thanks to the regularization, the r.h.s. of (4.33) and (4.34) are Lipschitz continuous. Hence, there is a solution $t \to \mathcal{X}(t)$ to the dynamics given by (4.33) - (4.36) which is unique given any initial condition, continuously differentiable in t, and Lipschitz continuous in the choice of initial condition.

REMARK: We will later be interested in rescalings (by N) of the various quantities above. So, a word on units is in order here. We have assumed the existence of some *a priori* units for the basic physical quantities t, q and p (with the restriction that $c = 1$ in these units). This system of units could be natural "microscopic" units (determined in some fashion by the particles under study), some "macroscopic" system of units (determined perhaps by experimental convention), etc. The key thing is that these base units are *independent* of the number of particles, N.

4.4.2 Empirical Measure Description

As in the previous section, it is physically more informative to recast the dynamics given above in terms of empirical measures instead of thinking of our system as an abstract point in $\mathbb{R}^{6N}$.

We make the following definitions:

$$ {}^N\Delta_t^+(p,q) = \frac{1}{N}\sum_{i=1}^{N} \delta(p - p_{2i}(t))\delta(q - q_{2i}(t)), \tag{4.37} $$

$$ {}^N\rho_t^+(q) = \frac{1}{N}\sum_{i=1}^{N} \delta(q - q_{2i}(t)), \tag{4.38} $$

$$ {}^N\Delta_t^-(p,q) = \frac{1}{N}\sum_{i=1}^{N} \delta(p - p_{2i-1}(t))\delta(q - q_{2i-1}(t)), \tag{4.39} $$

$$ {}^N\rho_t^-(q) = \frac{1}{N}\sum_{i=1}^{N} \delta(q - q_{2i-1}(t)), \tag{4.40} $$

Note that ${}^N\rho_t^\pm$ are just the normalized spatial distributions of the particle species:

$$ {}^N\rho_t^\pm(q) = \int {}^N\Delta_t^\pm(p,q)d^3p. $$

Once we factor out the arbitrary particle labeling, these singular measures are in one-to-one correspondence with the phase space configurations of our system. With these, we can express our dynamics (4.33) - (4.36) as a coupled pair of PDEs for the densities (4.37) and (4.39):

$$\partial_t {}^N\!\Delta_t^{\pm} + v(p) \cdot \nabla_q {}^N\!\Delta_t^{\pm} \pm N G_\epsilon * \left[{}^N\!\rho_t^{+} - {}^N\!\rho_t^{-}\right](q) \cdot \nabla_p {}^N\!\Delta_t^{\pm} = 0, \tag{4.41}$$

where G_ϵ was defined by (4.1).

4.4.3 Initial Conditions

We assume the initial condition

$$\mathcal{X}(0) \equiv (q_1(0), p_1(0), \ldots, q_{2N}(0), p_{2N}(0)) \in \mathbb{R}^{12N}$$

for our plasma is chosen randomly according to

$$\mathbb{P}_0 = \bigotimes_{i=1}^{2N} f_0 \, d^3p \, d^3q, \tag{4.42}$$

where $f_0 \geq 0$ is any sufficiently nice function (continuously differentiable, say) on $\mathbb{R}^6$ with

$$\iint f_0(p,q) \, d^3p \, d^3q = 1.$$

In other words, we assume that each particle is initially distributed *iid* in $\mathbb{R}^6$ according to $f_0 \; d^3p \, d^3q$. Note that our densities ${}^N\!\Delta_\cdot^{\pm}$ are formally probability measures as well. Hence, we can use the dual bounded Lipschitz distance, d_{bL^*}, as in the previous section to measure how close our discrete initial data are to f_0.

Our assumption that each particle is initially distributed *iid* by $f_0 \; d^3p \, d^3q$ ensures

$$\lim_{N\to\infty} \mathbb{E}_{\mathbb{P}_0} \left[d_{bL^*}({}^N\!\Delta_0^{\pm}, f_0) \right] = 0.$$

In fact, we can say a great deal more! Namely, the theory of large deviations supplies a rate

$$\mathbb{P}_0[d_{bL^*}({}^N\!\Delta_0^{\pm}, f_0) > \delta] \asymp e^{-\underline{\mathcal{H}}_{f_0}(\delta) N}$$

where the rate function $\underline{\mathcal{H}}_{f_0}$ depends only on f_0 (this is explained in detail in the next section).

4.5 The Large Deviation Principle and Entropy Estimates for the Initial Conditions

4.5.1 The Large Deviation Principle

We first study the expected rate of convergence under our assumption that the particles are initially distributed *iid* according to $f_0 d^3p d^3q$. The mainstay in such situations is the Theory of Large Deviations. For an excellent overview of this topic, the reader is referred to the review article by Touchette [To09]. For a more detailed analysis, see [DZ98].

Our initial empirical measures ${}^N\Delta_0^{\pm}$ are random elements of $M_1(\mathbb{R}^6)$ (the set of probability measures on $\mathbb{R}^6$) chosen according to $\mathbb{P}_0$ defined in (4.42). Sanov's Theorem (c.f. Theorem 6.2.10 of [DZ98]) states that these measures satisfy a large deviation principle in $M_1(\mathbb{R}^6)$ equipped with the so-called τ–topology (the topology generated by all bounded, Borel measurable functions on $\mathbb{R}^6$). Moreover, the rate function is given by the relative entropy $H(\cdot|f_0 d^3p d^3q)$ where for $\mu, \nu \in M_1(\mathbb{R}^6)$

$$H(\mu|\nu) \equiv \int \ln\left(\frac{d\mu}{d\nu}\right) d\mu = \int \frac{d\mu}{d\nu} \ln\left(\frac{d\mu}{d\nu}\right) d\nu \tag{4.43}$$

if the Radon-Nikodym derivative of μ w.r.t. ν exists; otherwise, $H(\mu|\nu) = \infty$. Thus, for any measurable set A on $M_1(\mathbb{R}^6)$

$$\mathbb{P}_0\left({}^N\Delta_0^{\pm} \in A\right) \asymp \exp\left(-\inf_{\mu\in A}\{H(\mu|f_0 d^3p d^3q)\}N\right).$$

This rate function also works in the topology given by the dual, bounded Lipschitz metric, as it is a far weaker topology than the τ-topology. To give a precise statement, we appeal to the *Contraction Principle* (c.f. Theorem 4.2.1 of [DZ98]). Since $d_{bL^*}({}^N\Delta_0^{\pm}, f_0)$ is a continuous function of the random variable ${}^N\Delta_0^{\pm}$, this principle states that the metric also satisfies a large deviation principle with rate function

$$\underline{\mathcal{H}}_{f_0}(\delta) \equiv \inf_{\mu\in M_1(\mathbb{R}^6)}\{H(\mu|f_0 d^3p d^3q) \; : \; d_{bL^*}(\mu, f_0) = \delta\} \tag{4.44}$$

for any $\delta > 0$.

4.5.2 Entropy Estimates

To make further progress, we need to make some assumption about f_0. We outline several possibilities (taken from [Vi03] and [BGL01]) that lead to definite estimates for $\underline{\mathcal{H}}_{f_0}(\delta)$.

The first inequality we will use was first proven by Otto and Villani (see Chapter 9 of [Vi03] for more details). Suppose there is some $\lambda > 0$ such that $f_0 d^3p d^3q$ satisfies

$$H(\mu|f_0 d^3p d^3q) \quad \leq \quad \frac{1}{2\lambda} I(\mu|f_0 d^3p d^3q) \tag{4.45}$$

for all $\mu \in M_1(\mathbb{R}^6)$ where I is the relative Fisher Information

$$I(\mu|\nu) \equiv \int \left| \nabla \left(\ln \frac{d\mu}{d\nu} \right) \right|^2 d\mu.$$

Then f_0 is said to satisfy a logarithmic Sobolev Inequality with constant λ (abbreviated LSI(λ) in [Vi03]). There are no easy characterizations for f_0 that are equivalent to this property (at least in dimensions greater than 1). It is known (c.f. [Vi03] Theorem 9.9) that if

$$\nabla^2 \ln \frac{1}{f_0} \geq \lambda \mathbb{I}_6$$

then f_0 satisfies LSI(λ).

If f_0 satisfies LSI(λ), then according to (9.41) in [Vi03] f_0 satisfies a so-called *Talagrand Inequality* (labeled T(λ) in the reference). Namely, for all $\mu \in M_1(\mathbb{R}^6)$

$$W_2(\mu, f_0 d^3p d^3q) \leq \sqrt{\frac{2}{\lambda} H(\mu|f_0 d^3p d^3q)}, \tag{4.46}$$

where W_2 is the Wasserstein metric with quadratic cost. In general, for $p \geq 1$ we have

$$W_p(\mu, \nu) \quad = \quad \left(\sup_{\pi \in \Pi(\mu,\nu)} \int |x-y|^p d\pi(x,y) \right)^{1/p}$$

where $\Pi(\mu, \nu)$ is the set of all probability measures $\pi \in M_1(\mathbb{R}^6 \times \mathbb{R}^6)$ such that for all measurable $A \in \mathbb{R}^6$

$$\pi(A \times \mathbb{R}^6) = \mu(A) \ \text{ and } \ \pi(\mathbb{R}^6 \times A) = \nu(A).$$

Kantorovich-Rubinstein duality (together with Jensen's Inequality) implies

$$d_{bL^*}(\mu, \nu) \leq W_1(\mu, \nu) \leq W_p(\mu, \nu)$$

for all $p > 1$ (see Chapter 1 and Remarks 7.5 in [Vi03]).

In our case, we have that if f_0 satisfies LSI(λ) then for all probability measures $\mu \in M_1(\mathbb{R}^6)$

$$H(\mu|f_0 d^3p d^3q) \geq \frac{\lambda}{2} \left(d_{bL^*}(\mu, f_0)\right)^2. \tag{4.47}$$

A natural question is whether we can improve on these results by demanding f_0 satisfy a more stringent inequality than LSI(λ). The answer seems to be no, as we outline below.

We pick up on ideas outlined in [BGL01]. In this work, the authors recover the result LSI(λ) $\rightarrow$ T(λ) through a dual formulation (essentially Kantorovich-Rubinstein duality) paired with results on Hamilton-Jacobi equations. Though the work strongly concentrates on the quadratic cost (i.e. the metric W_2), section 5 of [BGL01] considers the situation for non-quadratic costs. We adapt their results to our present circumstances. We will give a sketch of their derivation below for those whom these results are unfamiliar.

First, as noted above

$$d_{bL^*}(\mu, \nu) \leq W_1(\mu, \nu) \leq W_{1+\kappa}(\mu, \nu)$$

for any probability measures μ, ν and any $\kappa > 0$. We wish to find conditions on μ so that

$$W_{1+\kappa}(\mu, \nu)^{1+\kappa} \quad \leq \quad \frac{1+\kappa}{\lambda} H(\nu|\mu) \tag{4.48}$$

for all probability measures ν. Clearly, we only need to worry about those ν which are absolutely continuous w.r.t μ (as otherwise the relative entropy is $+\infty$).

First, the Kantorovich-Rubinstein duality theorem gives

$$W_{1+\kappa}(\mu, \nu)^{1+\kappa} \quad = \quad \sup \left\{ \int g d\nu - \int f d\mu \ : \ g(x) \leq f(y) + |x-y|^{1+\kappa} \right\}$$

where the supremum is taken over all pairs of bounded measurable functions (f, g) satisfying the stated condition. Hence for any particular admissible pair, (4.48) implies

$$\int g d\nu - \int f d\mu \leq \frac{1+\kappa}{\lambda} H(\nu|\mu).$$

In particular, we can take

$$g(x) = \inf_y \left\{ f(y) + |x-y|^{1+\kappa} \right\} \equiv Qf(x).$$

Taking $d\nu = \phi d\mu$ (where of course $\int \phi d\mu = 1$) we get

$$\begin{aligned} H(\nu|\mu) &\geq \frac{\lambda}{1+\kappa} \int Qf \, \phi d\mu - \int f d\mu \\ &\geq \frac{\lambda}{1+\kappa} \int \left(Qf - \int f d\mu \right) \phi d\mu. \end{aligned}$$

Since this inequality is to hold for any choice of ν (equivalently, any choice of ϕ with $\int \phi d\mu = 1$) we can choose

$$\phi_0 \equiv \frac{e^{\psi}}{\int e^{\psi} d\mu}$$

where $\psi \equiv \frac{\lambda}{1+\kappa} \left(Qf - \int f d\mu \right)$. Plugging this choice into the above inequality and simplifying the result gives $\int e^{\psi} d\mu \leq 1$ which means

$$\int e^{\frac{\lambda}{1+\kappa} Qf} d\mu \quad \leq \quad e^{\frac{\lambda}{1+\kappa} \int f d\mu} \tag{4.49}$$

which should hold for every bounded measurable function f. We find it convenient to absorb the factor $1/1+\kappa$ into f (as f ranges over all bounded measurable functions). Defining

$$Q_1^{\kappa} f(x) \quad \equiv \quad \inf_y \left\{ f(y) + \frac{|x-y|^{1+\kappa}}{1+\kappa} \right\} \tag{4.50}$$

we get the following inequality which should hold for all bounded measurable f:

$$\int e^{\lambda Q_1^{\kappa} f} d\mu \quad \leq \quad e^{\lambda \int f d\mu} \tag{4.51}$$

(though the choice of superscript on Q is obvious, the subscript "1" will only become clear below).

Now suppose (4.51) holds for every bounded measurable function f. Of course, we can immediately revert back to (4.49) by pulling out a factor of $1/1+\kappa$. It is well known that

$$H(\nu|\mu) = \sup \left\{ \int h d\nu \ : \ \int e^h d\mu \leq 1 \right\}.$$

Thus,

$$\begin{aligned} H(\nu|\mu) &\geq \int \psi d\nu \\ &\geq \frac{\lambda}{1+\kappa}\left(\int Qf d\nu - \int f d\mu\right) \end{aligned}$$

Clearly if g is any bounded measurable function satisfying

$$g(x) \leq f(y) + |x-y|^{1+\kappa}$$

for all x and y, then $g \leq Qf$. Hence,

$$H(\nu|\mu) \geq \frac{\lambda}{1+\kappa}\left(\int g d\nu - \int f d\mu\right)$$

for any pair (f,g) satisfying $g(x) \leq f(y) + |x-y|^{1+\kappa}$ for all x and y. Taking the supremum over all such pairs shows that inequalities (4.48) and (4.51) are actually equivalent.

To proceed, we need only find an equivalent inequality to (4.51). To accomplish this, we cite some facts about Hamilton-Jacobi equations from section 5 of [BGL01]. Let $\mathcal{H}$ be smooth and convex on $\mathbb{R}^n$ with $\lim_{|x|\to\infty} \mathcal{H}(x)/|x| = +\infty$. Consider the Hamilton-Jacobi initial value problem:

$$\begin{cases} \frac{\partial u}{\partial t} + \mathcal{H}(\nabla u) = 0 & \text{in } \mathbb{R}^n \times (0,\infty) \\ u = f & \text{on } \mathbb{R}^n \times \{t=0\} \end{cases}$$

where f is a sufficiently nice function (Lipschitz continuous will do, for example). In a suitably weak sense, this PDE is solved by the following (often termed a viscosity solution)

$$u(x,t) \equiv \begin{cases} Q_t^{\mathcal{L}} f(x) \equiv \inf_y \left\{ f(y) + t\mathcal{L}\left(\frac{x-y}{t}\right)\right\} & \text{for } t > 0 \\ f(x) & \text{for } t = 0 \end{cases}$$

where $\mathcal{L}$ is the convex conjugate of $\mathcal{H}$. For arbitrary $\mathcal{H}$, u will not be continuous at $t=0$, which is a property that we will require. Fortunately, we will take

$$\mathcal{L}(x) = \frac{|x|^{1+\kappa}}{1+\kappa}$$

and so

$$H(y) = \frac{|y|^{1+1/\kappa}}{1+1/\kappa}.$$

Specializing to this case gives

$$Q_t^\kappa f(x) \equiv \inf_y \left\{ f(y) + \frac{|x-y|^{1+\kappa}}{(1+\kappa)t^\kappa} \right\}$$

(which finally vindicates our previous notation). We will need

$$\lim_{t \searrow 0} Q_t^\kappa f(x) = f(x)$$

which is true if and only if f is lower semicontinuous at x. This is not a major issue since we can approximate any bounded, measurable function arbitrarily well by a lower semicontinuous function (by convolving f with some suitably nice function, for instance). Hence, we can take the continuity requirement for granted.

We now consider the function

$$\varphi_f^\kappa(t) \equiv \frac{1}{t} \ln \left(\int e^{\lambda t Q_t^\kappa f} d\mu \right).$$

By the continuity of $Q_t^\kappa f$ at $t = 0$ we have

$$\varphi_f^\kappa(0) = \lambda \int f d\mu.$$

Clearly (4.51) is equivalent to the requirement that

$$\varphi_f^\kappa(1) \leq \lambda \int f d\mu.$$

So, we need $\varphi_f^\kappa(t)$ to be non-increasing for $t \in (0,1]$ for every choice of f. Taking the derivative gives

$$\begin{aligned} \frac{d}{dt}\varphi_f^\kappa(t) = &- \frac{1}{t^2} \ln \left(\int e^{\lambda t Q_t^\kappa f} d\mu \right) \\ &+ \frac{1}{t} \left(\int e^{\lambda t Q_t^\kappa f} d\mu \right)^{-1} \int \left(\lambda Q_t^\kappa f + \lambda t \frac{\partial}{\partial t} Q_t^\kappa f \right) e^{\lambda t Q_t^\kappa f} d\mu. \end{aligned}$$

Forcing this derivative to be non-positive, using the Hamilton-Jacobi equation, and rearranging the terms yields the equivalent inequality

$$\begin{aligned} \lambda t \int Q_t^\kappa f e^{\lambda t Q_t^\kappa f} d\mu - & \left(\int e^{\lambda t Q_t^\kappa f} d\mu \right) \ln \left(\int e^{\lambda t Q_t^\kappa f} d\mu \right) \\ & \leq \frac{1}{(1+1/\kappa)\lambda^{1/\kappa} t^{1/\kappa - 1}} \int |\nabla \lambda t Q_t^\kappa f|^{1+1/\kappa} e^{\lambda t Q_t^\kappa f} d\mu. \end{aligned}$$

We will take $0 < \kappa \le 1$ and $0 < t \le 1$. Hence $t^{1-1/\kappa} \ge 1$. Therefore, the above inequality will be satisfied as long as

$$\int \lambda t Q_t^\kappa f e^{\lambda t Q_t^\kappa f} d\mu - \left(\int e^{\lambda t Q_t^\kappa f} d\mu\right) \ln \left(\int e^{\lambda t Q_t^\kappa f} d\mu\right) \le \frac{1}{(1+1/\kappa)\lambda^{1/\kappa}} \int |\nabla \lambda t Q_t^\kappa f|^{1+1/\kappa} e^{\lambda t Q_t^\kappa f} d\mu \tag{4.52}$$

for all appropriate f and all $t \in (0,1]$. Replacing $\lambda t Q_t^\kappa f$ by a generic (sufficiently smooth) function g gives the inequality

$$\int g e^g d\mu - \left(\int e^g d\mu\right) \ln \left(\int e^g d\mu\right) \le \frac{1}{(1+1/\kappa)\lambda^{1/\kappa}} \int |\nabla g|^{1+1/\kappa} e^g d\mu.$$

Defining the measure

$$d\nu = \frac{e^g}{\int e^g d\mu} d\mu$$

gives the above inequality in the form

$$H(\nu|\mu) \le \frac{1}{(1+1/\kappa)\lambda^{1/\kappa}} \int \left|\nabla \ln \frac{d\nu}{d\mu}\right|^{1+1/\kappa} d\nu \tag{4.53}$$

which is to hold for any choice of probability measure ν absolutely continuous w.r.t. μ. We refer to (4.53) by saying μ satisfies LSI(κ, λ).

Finally, our considerations above show that if μ satisfies LSI(κ, λ), then for all probability measures ν, we have

$$\frac{\lambda}{1+\kappa} W_{1+\kappa}(\mu, \nu)^{1+\kappa} \quad \le \quad H(\nu|\mu),$$

and so by considerations mentioned above

$$H(\nu|\mu) \ge \frac{\lambda}{1+\kappa} \left(d_{bL^*}(\mu, \nu)\right)^{1+\kappa}. \tag{4.54}$$

Some comments are in order at this point. Taking a trial density of the form

$$d\nu = \frac{(1+\beta g) d\mu}{1 + \beta \int g d\mu}$$

for some sufficiently nice, non-negative function g and non-negative constant β, inserting this into LSI(κ, λ), and looking at the asymptotics for small β gives that

$$H(\nu|\mu) \asymp \beta^2$$

while

$$\frac{1}{(1+1/\kappa)\lambda^{1/\kappa}} \int \left| \nabla \ln \frac{d\nu}{d\mu} \right|^{1+1/\kappa} d\nu \asymp \beta^{1+1/\kappa}.$$

Whenever $\kappa < 1$, we get a contradiction to LSI(κ, λ) by taking β sufficiently small. Thus, for μ to satisfy LSI(κ, λ) we must have $\kappa \geq 1$.

In general, there is no known simpler condition on μ which is equivalent to LSI(κ, λ) when $\kappa \geq 1$. However, there are some results on sufficient conditions. The Bakry-Emery condition (namely

$$\nabla^2 \left(-\ln \frac{d\mu}{dx} \right) \geq \lambda \mathbb{I}_6$$

mentioned above) is sufficient for μ (which is assumed to be absolutely continuous w.r.t. Lebesgue measure) to satisfy LSI$(1, \lambda)$. For general LSI(κ, λ) with $\kappa \geq 1$, we cite the work of Bobkov and Ledoux [BL00] wherein the authors find a sufficient condition based on the Brunn-Minkowski Inequality.

Suppose that $d\mu = \exp(-V)dx$ for some convex function V. If V satisfies the condition

$$V(x) + V(y) - 2V\left(\frac{x+y}{2}\right) \quad \geq \quad \frac{\lambda}{\kappa}\|x-y\|^{1+\kappa}, \tag{4.55}$$

for all x and y and $\kappa \geq 1$, then μ satisfies LSI(κ, λ). We note that this condition gives further evidence that the $\kappa < 1$ case is problematic! Choosing x to be $x+h$ and y to be $x-h$ gives

$$\frac{V(x+h) - 2V(x) - V(x-h)}{\|h\|^2} \quad \geq \quad \frac{\lambda}{\kappa}\|h\|^{\kappa-1}.$$

Taking the limit as $\|h\| \to 0$ shows us that any such function must have unbounded (symmetric) second derivative for any x. Since a finite convex function has second derivative almost everywhere (by Alexandroff's Theorem c.f. [Ro99]), we get a contradiction.

4.5.3 Rates of Convergence for Initial Conditions

If $f_0 d^3p d^3q$ satisfies LSI(κ, λ) then for all $\delta > 0$

$$\underline{\mathcal{H}}_{f_0}(\delta) \equiv \inf_{\mu \in M_1(\mathbb{R}^6)} \{ H(\mu | f_0 d^3p d^3q) \ : \ d_{bL^*}(\mu, f_0) = \delta \} \geq \frac{\lambda \delta^{1+\kappa}}{1+\kappa} \tag{4.56}$$

and so

$$\mathbb{P}_0\left(d_{bL^*}({}^N\Delta_0^{\pm}, f_0) > \delta\right) \lesssim \exp\left(-\frac{\lambda\delta^{1+\kappa}}{1+\kappa}N\right) \tag{4.57}$$

which shows that at least at $t = 0$, $d_{bL^*}({}^N\Delta_0^{\pm}, f_0)$ converges to zero in probability.

Under this assumption, we can also estimate the expected value of $d_{bL^*}({}^N\Delta_0^{\pm}, f_0)$:

$$\mathbb{E}_{\mathbb{P}_0}\left[d_{bL^*}({}^N\Delta_0^{\pm}, f_0)\right] \lesssim \int_0^{\infty} \delta \exp\left(-\frac{\lambda\delta^{1+\kappa}}{1+\kappa}N\right) d\delta$$

for sufficiently large N. Therefore, if $f_0 d^3p d^3q$ satisfies LSI(κ, λ) we have

$$\mathbb{E}_{\mathbb{P}_0}\left[d_{bL^*}({}^N\Delta_0^{\pm}, f_0)\right] \lesssim \left(\frac{1}{\lambda N}\right)^{\frac{2}{1+\kappa}} \Gamma\left(\frac{2}{1+\kappa}\right)(1+\kappa)^{\frac{1-\kappa}{1+\kappa}}. \tag{4.58}$$

Unfortunately, the best we can do with the results in hand is take $\kappa = 1$. Hence,

$$\mathbb{E}_{\mathbb{P}_0}\left[d_{bL^*}({}^N\Delta_0^{\pm}, f_0)\right] \lesssim \frac{\mathcal{C}}{N}.$$

As we will see, this rate is sufficient for showing the emergence of rVP$_\epsilon^-$ on short time scales. However, the lack of a stronger rate of convergence will have severe drawbacks.

4.6 Space-time Rescalings

With separate initial profiles for the individual species (i.e. the positive and negative particles are chosen *iid* by two different probability measures), the force term (4.34) typically grows on the order of N. This effectively destroys any limiting behavior *on the a priori space-time scale*. This is also the case for single specie models typically encountered in Vlasov limits (such as the one considered in §4.3). Many ways have been proposed to avoid this. The most common method in non-relativistic situations is to assume the mass of the particles is N^{-1}. This has several problems. First and foremost, it is not feasible in the relativistic setting. Sending the mass of the particles to zero will send the velocities of the particles to the speed of light in the limit! This is certainly not a desirable state of affairs. Even in the non-relativistic setting where rescaling the masses works, it is an unphysical procedure. The primary objects we are attempting to model (be they fundamental particles, atoms, stars, etc.) have some fixed, natural mass - end of story. Of course, if we understand this as a *rescaling* of the units of mass, then there is no conceptual problem. We find that the more acceptable procedure is to rescale the space and time units (and associated parameters) in the model in such a way that the force term (4.34) no longer grows with the number of particles, but remains of order 1 regardless of the number of particles. To that end, we introduce the following rescaled variables:

$$\bar{t} = N^{\alpha} t, \tag{4.59}$$

$$\bar{q} = N^{\alpha} q, \tag{4.60}$$

$$\bar{\epsilon} = N^{\alpha} \epsilon, \tag{4.61}$$

$$\bar{p} = p. \tag{4.62}$$

A few comments are in order before determining the equations governing the system in the barred coordinates. First, (4.62) is forced upon us by the relativistic velocity; i.e. the speed of light (1 in our units) is a universal speed limit! Since the l.h.s of (4.33) does not rescale as we vary N, we must ensure dq/dt is independent of N in the rescaling (otherwise the velocities could limit to zero or infinity). Hence, $\bar{t}$ and $\bar{q}$ must be of the same order in N. The choice for ϵ is more subtle (mainly motivated by the desire to keep formulae tractable). We could consider more general rescalings; for instance, we could allow the speed of light *as measured in our units* to change with N (thus allowing space and time to be scaled independently). However, the choices detailed above are sufficient for our purposes.

First, we must insist that rescaled densities ${}^N\bar{\Delta}^{\pm}_{\bar{t}}$ continue to have total mass 1.

$$\begin{aligned} 1 &= \iint {}^N\Delta^{\pm}_{t}(p,q) d^3p d^3q \\ &= \iint {}^N\Delta^{\pm}_{N^{-\alpha}\bar{t}} \left(\bar{p}, N^{-\alpha}\bar{q}\right) N^{-3\alpha} d^3\bar{p} d^3\bar{q}. \end{aligned}$$

So, we define

$${}^N\bar{\Delta}^{\pm}_{\bar{t}}(\bar{p},\bar{q}) = N^{-3\alpha} \;\; {}^N\Delta^{\pm}_{N^{-\alpha}\bar{t}} \left(\bar{p}, N^{-\alpha}\bar{q}\right). \tag{4.63}$$

Next, we note that from our choice of $\bar{\epsilon}$

$$\begin{aligned} \eta_\epsilon(q) &= \frac{1}{\epsilon^3}\eta\left(\frac{q}{\epsilon}\right) \\ &= \frac{N^{3\alpha}}{\bar{\epsilon}^3}\eta\left(\frac{\bar{q}}{\bar{\epsilon}}\right) \\ &= N^{3\alpha}\eta_{\bar{\epsilon}}(\bar{q}). \end{aligned}$$

Note that this preserves the integral

$$\int \eta_\epsilon(q)d^3q = \int \eta_{\bar{\epsilon}}(\bar{q})d^3\bar{q} = 1$$

so that η_ϵ and $\eta_{\bar{\epsilon}}$ are approximate identities (as ϵ and $\bar{\epsilon}$ go to zero) on their respective scales. This gives

$$G_\epsilon(q_1 - q_2) = N^{2\alpha}G_{\bar{\epsilon}}(\bar{q}_1 - \bar{q}_2).$$

So, the force term becomes

$$NG_\epsilon * \left[{}^N\!\rho_t^+ - {}^N\!\rho_t^-\right](q) = N^{1+2\alpha}G_{\bar{\epsilon}} * \left[{}^N\!\bar{\rho}_t^+ - {}^N\!\bar{\rho}_t^-\right](\bar{q}).$$

Finally by taking into account the correct factors in the t and q derivatives, we get the PDE governing ${}^N\bar{\Delta}_{\bar{t}}^{\pm}$:

$$\partial_{\bar{t}}{}^N\bar{\Delta}_{\bar{t}}^{\pm} + v(\bar{p})\cdot\nabla_{\bar{q}}{}^N\bar{\Delta}_{\bar{t}}^{\pm} \pm N^{1+\alpha}G_{\bar{\epsilon}} * \left[{}^N\!\bar{\rho}_t^+ - {}^N\!\bar{\rho}_t^-\right](\bar{q})\cdot\nabla_{\bar{p}}{}^N\bar{\Delta}_{\bar{t}}^{\pm} = 0. \tag{4.64}$$

The scales of interest for us will be $\alpha = 0$ which corresponds to the *a priori* space-time scale and $\alpha = -1$ which removes the factor of N from the force term of (4.64) altogether. The choice of $\alpha = -1$ corresponds to $\bar{t}$ and $\bar{q}$ being of order N larger than the underlying scales t and q (that is to long-time and large-space scales). This is the traditional scale where one pursues Vlasov limits. In what follows we shall refer to the scaling $\alpha = -1$ as the traditional Vlasov scale.

4.7 The Traditional Vlasov Space-Time Scale

We will begin by exploring the behavior of our plasma model on the traditional Vlasov scale ($\alpha = -1$ in the terminology above). Since we will stay on this scale throughout the section, we drop all bars on the various objects in question. To be clear, the PDEs satisfied by our plasma are

$$\partial_t {}^N\Delta_t^{\pm} + v(p) \cdot \nabla_q {}^N\Delta_t^{\pm} \pm G_\epsilon * \left[{}^N\rho_t^{+} - {}^N\rho_t^{-}\right](q) \cdot \nabla_p {}^N\Delta_t^{\pm} = 0. \tag{4.65}$$

We shall show that on this scale, the infinitely many particles limit with initial condition chosen *iid* by f_0 for all particles is to a free-streaming fluid for both species:

$$\partial_t f_t + v(p) \cdot \nabla_q f_t = 0, \tag{4.66}$$

with initial condition f_0. The presentation here will mirror that of §4.3.

Actually, since on this space-time scale we do not have to worry about extra factors of N (which will greatly complicate matters on the *a priori* space-time scales) we can proceed a bit more generally. Hence, we will assume that the positive and negative particles are still chosen independently, but we will drop the assumption that they are chosen identically. Specifically, we assume that the N positive particles are chosen *iid* by some f_0^+, and the N negative particles are chosen *iid* by some f_0^-. We still have the discrete dynamics given by (4.65), but the infinitely many particles limit will be to a set of coupled PDEs (denoted hereafter as the *two-specie relativistic Vlasov-Poisson system*):

$$\partial_t f_t^{\pm} + v(p) \cdot \nabla_q f_t^{\pm} \pm G_\epsilon * \left[\rho_{f_t^+} - \rho_{f_t^-}\right](q) \cdot \nabla_p f_t^{\pm} = 0, \tag{4.67}$$

where

$$\rho_{f_t^{\pm}}(q) \equiv \int f_t^{\pm}(p,q) d^3p. \tag{4.68}$$

Note that Sanov's Theorem still applies, and so we still have

$$\mathbb{E}_{\mathbb{P}_0}\left[d_{bL^*}({}^N\Delta_0^{\pm}, f_0^{\pm})\right] \quad \lesssim \quad \left(\frac{1}{\lambda N}\right)^{\frac{2}{1+\kappa}} \Gamma\left(\frac{2}{1+\kappa}\right)(1+\kappa)^{\frac{1-\kappa}{1+\kappa}},$$

when both $f_0^{\pm} d^3p d^3q$ satisfy LSI(κ, λ).

4.7.1 The Fixed Point Characterization

Associated to the PDEs (4.65) and (4.67) are vector fields on $\mathbb{R}^6$ given by

$$V^{\pm}\left[{}^N\Delta_{\cdot}^+, {}^N\Delta_{\cdot}^-\right](t,p,q) = \begin{bmatrix} \pm G_\epsilon * \left[{}^N\rho_t^+ - {}^N\rho_t^-\right](q) \\ v(p) \end{bmatrix} \tag{4.69}$$

$$V^{\pm}\left[f_{\cdot}^+, f_{\cdot}^-\right](t,p,q) = \begin{bmatrix} \pm G_\epsilon * \left[\rho_{f_t^+} - \rho_{f_t^-}\right](q) \\ v(p) \end{bmatrix} \tag{4.70}$$

$$\tag{4.71}$$

(we have written the vector fields with the same symbol since they have the same functional form). Associated to these fields are the induced flows on $\mathbb{R}^6$:

$$T_{t,0}^{\pm}\left[{}^N\Delta_{\cdot}^+, {}^N\Delta_{\cdot}^-\right](p,q) \quad \text{and} \quad T_{t,0}^{\pm}\left[f_{\cdot}^+, f_{\cdot}^-\right](p,q),$$

Both of these flows are characterized as follows:

$$\begin{aligned} T_{t,0}^{\pm}\left[g_{\cdot}, h_{\cdot}\right](p,q) &= (p(t), q(t)), \\ \begin{pmatrix} \dot{p}_i(t) \\ \dot{q}_i(t) \end{pmatrix} &= V^{\pm}\left[g_{\cdot}, h_{\cdot}\right](t, p(t), q(t)), \\ (p(0), q(0)) &= (p,q),. \end{aligned} \tag{4.72}$$

As before, these flows preserve phase space volume in $\mathbb{R}^6$. Having established the relevant definitions, we can now see that our PDEs (4.65) and (4.67) are equivalent to:

$${}^N\Delta_t^{\pm}(p,q) = {}^N\Delta_0^{\pm} \circ T_{0,t}^{\pm}\left[{}^N\Delta_{\cdot}^+, {}^N\Delta_{\cdot}^-\right](p,q), \tag{4.73}$$

$$f_t^{\pm}(p,q) = f_0^{\pm} \circ T_{0,t}^{\pm}\left[f_{\cdot}^+, f_{\cdot}^-\right](p,q). \tag{4.74}$$

4.7.2 Lipschitz Constants

We first establish Lipschitz constants for our vector fields and flows. Thanks to our considerations in §4.2 we have:

$$\begin{aligned} \left| \int \left(G_\epsilon(q_1 - q') \right.\right. & \left.\left. - G_\epsilon(q_2 - q')\right)\left[\rho_{f_t^+} - \rho_{f_t^-}\right] d^3q' \right| \\ & \leq L_G |q_1 - q_2| \int \rho_{f_t^+}(q') + \rho_{f_t^-}(q') d^3q' \\ & \leq 2L_G |q_1 - q_2| \end{aligned}$$

Using (4.8), we get a Lipschitz constant for the charged vector fields and flows:

$$\left|V^{\pm}\left[f_\cdot^+, f_\cdot^-\right](t, p_1, q_1) - V^{\pm}\left[f_\cdot^+, f_\cdot^-\right](t, p_2, q_2)\right| \le L_{\pm}|(p_1, q_1) - (p_2, q_2)|, \tag{4.75}$$

and by an application of Gronwall's Inequality:

$$\left|T_{t,0}^{\pm}\left[f_\cdot^+, f_\cdot^-\right](p_1, q_1) - T_{t,0}^{\pm}\left[f_\cdot^+, f_\cdot^-\right](p_2, q_2)\right| \le e^{L_{\pm}t}|(p_1, q_1) - (p_2, q_2)|, \tag{4.76}$$

where

$$L_{\pm} = \max\{1, 2L_G\}. \tag{4.77}$$

Note that these considerations would have worked just as well for the flows generated by the discrete probability measures ${}^N\Delta_t^{\pm}$.

4.7.3 Convergence to the Two-specie Relativistic Vlasov-Poisson System

We proceed much like §4.3. As usual, φ will denote an arbitrary element of $\mathcal{D}(\mathbb{R}^6)$ as defined in (4.25).

$$\begin{aligned}
&\left|\iint \varphi(p,q)\left({}^N\Delta_t^+(p,q) - f_t^+(p,q)\right) d^3p d^3q\right| \\
&\quad = \left|\iint \varphi(p,q)\left({}^N\Delta_0^+ \circ T_{0,t}^+\left[{}^N\Delta_\cdot^+, {}^N\Delta_\cdot^-\right](p,q) - f_0 \circ T_{0,t}^+\left[f_\cdot^+, f_\cdot^-\right](p,q)\right) d^3p d^3q\right| \\
&\quad \le \left|\iint \varphi(p,q)\left({}^N\Delta_0^+ \circ T_{0,t}^+\left[{}^N\Delta_\cdot^+, {}^N\Delta_\cdot^-\right](p,q) - {}^N\Delta_0^+ \circ T_{0,t}^+\left[f_\cdot^+, f_\cdot^-\right](p,q)\right) d^3p d^3q\right| \\
&\qquad + \left|\iint \varphi(p,q)\left({}^N\Delta_0^+ \circ T_{0,t}^+\left[f_\cdot^+, f_\cdot^-\right](p,q) - f_0 \circ T_{0,t}^+\left[f_\cdot^+, f_\cdot^-\right]\right) d^3p d^3q\right| \\
&\quad \le \iint \left|\varphi \circ T_{t,0}^+\left[{}^N\Delta_\cdot^+, {}^N\Delta_\cdot^-\right](p,q) - \varphi \circ T_{t,0}^+\left[f_\cdot^+, f_\cdot^-\right](p,q)\right| {}^N\Delta_0^+(p,q) d^3p d^3q \\
&\qquad + \left|\iint \varphi \circ T_{t,0}^+\left[f_\cdot^+, f_\cdot^-\right](p,q)\left({}^N\Delta_0^+(p,q) - f_0(p,q)\right) d^3p d^3q\right|
\end{aligned}$$

For the first term in this last inequality, we recall that φ is Lipschitz continuous with Lipschitz constant bounded by one (by the definition of $\mathcal{D}$). For the second term, we note that $e^{-L_{\pm}t}\varphi \circ T_{t,0}^+[f_\cdot^+, f_\cdot^-] \in \mathcal{D}(\mathbb{R}^6)$ by (4.76). Thus,

$$\begin{aligned}
d_{bL^*}({}^N\Delta_t^+, f_t^+) \le\; & e^{L_{\pm}t} d_{bL^*}({}^N\Delta_0^+, f_0^+) \\
& + \iint \left|T_{t,0}^+\left[{}^N\Delta_\cdot^+, {}^N\Delta_\cdot^-\right](p,q) - T_{t,0}^+\left[f_\cdot^+, f_\cdot^-\right](p,q)\right| {}^N\Delta_0^+(p,q) d^3p d^3q.
\end{aligned} \tag{4.78}$$

Since the first term is exactly the sort of bound we would like, we need only focus on the second term. We make the following definitions:

$$\lambda^+(t) = \iint \left| T^+_{t,0}\left[{}^N\Delta^+_\cdot, {}^N\Delta^-_\cdot\right](p,q) - T^+_{t,0}\left[f^+_\cdot, f^-_\cdot\right](p,q)\right| {}^N\Delta^+_0(p,q) d^3p d^3q, \quad (4.79)$$

$$\lambda^-(t) = \iint \left| T^-_{t,0}\left[{}^N\Delta^+_\cdot, {}^N\Delta^-_\cdot\right](p,q) - T^-_{t,0}\left[f^+_\cdot, f^-_\cdot\right](p,q)\right| {}^N\Delta^-_0(p,q) d^3p d^3q. \quad (4.80)$$

With these functions, we can write (thanks to charge symmetry)

$$d_{bL^*}({}^N\Delta^+_t, f^+_t) + d_{bL^*}({}^N\Delta^-_t, f^-_t) \le e^{L_\pm t}\left(d_{bL^*}({}^N\Delta^+_0, f^+_0) + d_{bL^*}({}^N\Delta^-_0, f^-_0)\right) + \lambda^+(t) + \lambda^-(t). \quad (4.81)$$

Iterating the flow gives

$$\begin{aligned}
&\left| T^+_{t,0}\left[{}^N\Delta^+_\cdot, {}^N\Delta^-_\cdot\right](p,q) - T^+_{t,0}\left[f^+_\cdot, f^-_\cdot\right](p,q)\right| \\
&\le \int_0^t \Big| V^+\left[{}^N\Delta^+_\cdot, {}^N\Delta^-_\cdot\right]\left(\tau, T^+_{\tau,0}\left[{}^N\Delta^+_\cdot, {}^N\Delta^-_\cdot\right](p,q)\right) \\
&\qquad - V^+\left[f^+_\cdot, f^-_\cdot\right]\left(\tau, T^+_{\tau,0}\left[f^+_\cdot, f^-_\cdot\right](p,q)\right)\Big| d\tau \\
&\le \int_0^t \Big| V^+\left[{}^N\Delta^+_\cdot, {}^N\Delta^-_\cdot\right]\left(\tau, T^+_{\tau,0}\left[{}^N\Delta^+_\cdot, {}^N\Delta^-_\cdot\right](p,q)\right) \\
&\qquad - V^+\left[f^+_\cdot, f^-_\cdot\right]\left(\tau, T^+_{\tau,0}\left[{}^N\Delta^+_\cdot, {}^N\Delta^-_\cdot\right](p,q)\right)\Big| d\tau \\
&\quad + \int_0^t \Big| V^+\left[f^+_\cdot, f^-_\cdot\right]\left(\tau, T^+_{\tau,0}\left[{}^N\Delta^+_\cdot, {}^N\Delta^-_\cdot\right](p,q)\right) \\
&\qquad - V^+\left[f^+_\cdot, f^-_\cdot\right]\left(\tau, T^+_{\tau,0}\left[f^+_\cdot, f^-_\cdot\right](p,q)\right)\Big| d\tau \\
&\le \int_0^t \Big| V^+\left[{}^N\Delta^+_\cdot, {}^N\Delta^-_\cdot\right]\left(\tau, T^+_{\tau,0}\left[{}^N\Delta^+_\cdot, {}^N\Delta^-_\cdot\right](p,q)\right) \\
&\qquad - V^+\left[f^+_\cdot, f^-_\cdot\right]\left(\tau, T^+_{\tau,0}\left[{}^N\Delta^+_\cdot, {}^N\Delta^-_\cdot\right](p,q)\right)\Big| d\tau \\
&\quad + L_\pm \int_0^t \left| T^+_{\tau,0}\left[{}^N\Delta^+_\cdot, {}^N\Delta^-_\cdot\right](p,q) - T^+_{\tau,0}\left[f^+_\cdot, f^-_\cdot\right](p,q)\right| d\tau
\end{aligned}$$

Define

$$\gamma^+(t) = \iint \left| V^+\left[{}^N\Delta^+_\cdot, {}^N\Delta^-_\cdot\right](t,p,q) - V^+\left[f^+_\cdot, f^-_\cdot\right](t,p,q)\right| {}^N\Delta^+_t(p,q) d^3p d^3q \quad (4.82)$$

(and similarly γ^-) which is obtained by integrating the integrand of the next to last line above against ${}^N\Delta^+_0(p,q)d^3pd^3q$, applying the obvious change of variables, and the fact that

$${}^N\Delta^+_t(p,q) = {}^N\Delta^+_0 \circ T^+_{0,t}\left[{}^N\Delta^+_\cdot, {}^N\Delta^-_\cdot\right](p,q).$$

Hence, we get the inequality

$$\lambda^+(t) \le L_\pm \int_0^t \lambda^+(\tau)d\tau + \int_0^t \gamma^+(\tau)d\tau,$$

and our simple corollary to Gronwall's Inequality (Theorem 4.3.1) gives

$$\lambda^+(t) \le e^{L_\pm t}\int_0^t \gamma^+(\tau)e^{-L_\pm \tau}d\tau. \tag{4.83}$$

We now turn our attention to $\gamma^+(t)$:

$$\begin{aligned}\gamma^+(t) &= \iint \left| G_\epsilon * \left[{}^N\!\rho_t^+ - {}^N\!\rho_t^- - \rho_{f_t^+} + \rho_{f_t^-}\right](q)\right| {}^N\!\Delta_t^+(p,q)d^3pd^3q \\ &\le \sqrt{3}L_G\left(d_{bL^*}\left({}^N\!\Delta_t^+, f_t^+\right) + d_{bL^*}\left({}^N\!\Delta_t^-, f_t^-\right)\right),\end{aligned}$$

where L_G is given by (4.10).

Combining this with (4.83) and the analogous results by charge symmetry gives

$$\lambda^+(t) + \lambda^-(t) \le 2\sqrt{3}L_G e^{L_\pm t}\int_0^t \left(d_{bL^*}\left({}^N\!\Delta_\tau^+, f_\tau^+\right) + d_{bL^*}\left({}^N\!\Delta_\tau^-, f_\tau^-\right)\right)e^{-L_\pm\tau}d\tau. \tag{4.84}$$

Combining this with (4.81) gives

$$\begin{aligned} e^{-L_\pm t}&\left(d_{bL^*}({}^N\!\Delta_t^+, f_t^+) + d_{bL^*}({}^N\!\Delta_t^-, f_t^-)\right) \\ &\le d_{bL^*}({}^N\!\Delta_0^+, f_0^+) + d_{bL^*}({}^N\!\Delta_0^-, f_0^-) \\ &\quad + 2\sqrt{3}L_G\int_0^t \left(d_{bL^*}\left({}^N\!\Delta_\tau^+, f_\tau^+\right) + d_{bL^*}\left({}^N\!\Delta_\tau^-, f_\tau^-\right)\right)e^{-L_\pm\tau}d\tau. \end{aligned} \tag{4.85}$$

Yet another application of Gronwall's Inequality gives the estimate

$$d_{bL^*}({}^N\!\Delta_t^+, f_t^+) + d_{bL^*}({}^N\!\Delta_t^-, f_t^-) \le \frac{e^{(2\sqrt{3}L_G + L_\pm)t}}{2\sqrt{3}L_G}\left(d_{bL^*}\left({}^N\!\Delta_0^+, f_0^+\right) + d_{bL^*}\left({}^N\!\Delta_0^-, f_0^-\right)\right). \tag{4.86}$$

Utilizing our results on Large Deviations from §4.5 shows us that

$$\mathbb{E}_{\mathbb{P}_0}\left[d_{bL^*}({}^N\!\Delta_t^+, f_t^+) + d_{bL^*}({}^N\!\Delta_t^-, f_t^-)\right] \lesssim \frac{e^{(2\sqrt{3}L_G + L_\pm)t}\Gamma\left(\frac{2}{1+\kappa}\right)(1+\kappa)^{\frac{1-\kappa}{1+\kappa}}}{\sqrt{3}L_G(\lambda N)^{\frac{2}{1+\kappa}}}. \tag{4.87}$$

For any $t \ge 0$, we can take N large enough to make the RHS above as small as we like. Hence, we get convergence to the coupled Vlasov system (4.67) in the infinitely many particles limit for all $t \ge 0$.

4.7.4 Remarks on Landau Damping

As we mentioned in the introduction, one of the striking phenomena of the *non-relativistic* Vlasov-Poisson system is Landau Damping. In 1946, Landau made a detailed study of the *linearized* Vlasov-Poisson system for plasmas near thermal equilibrium. He found that oscillations in a hot plasma (with periodic boundary conditions) are damped exponentially fast. Prior to this work, all models known to exhibit damping phenomena required an overall increase of entropy to do so (as with the Boltzmann equation). Since the damping of oscillations appears to represent an irreversible phenomenon in the system, it was widely assumed that time-reversible equations such as the Vlasov-Poisson system could not exhibit damping. The explanation for this apparent contradiction is that as vibrational modes in q-space are damped out, there is a corresponding increase in the vibrational modes in p-space. Hence, there is no increase of entropy.

These increasingly violent oscillations in p-space are difficult to capture in experimental settings. However in a beautiful series of experiments by Malmberg *et. al.* (see [Vi10] for the relevant references), the damping effect was observed, and through echo experiments the damping was shown not to be caused by dissipative effects.

For many years, no one had a mathematically definitive result to support the widely accepted idea that Landau Damping was a property of the Vlasov equations themselves and not an artefact of the linearization. Very recently, Mouhot and Villani [Vi10] succeeded in showing that Landau Damping is present in the Vlasov-Poisson system for a plasma in a torus.

For relativistic plasmas, the story is somewhat different. In 1994, Schlickeiser [Sch94] considered the relativistic Vlasov-Poisson equations for a hot isotropic plasma linearized about the appropriate equilibrium state (the Jüttner distribution — the analog of the Maxwell-Boltzmann distribution, adjusted to incorporate the universal speed limit imposed by special relativity). He found that for longitudinal waves in the plasma, oscillations with wave number below some critical constant k_c have superluminal phase velocity and do not exhibit any damping. Oscillations with wave number above k_c are still damped as in the non-relativistic case (though the rate of damping may be much different for k near k_c). An interesting problem would be to try and adapt the methods of Mouhot and Villani to the relativistic Vlasov-Poisson system and determine if this phenomenon is born out in the non-linear regime.

4.7.5 Convergence to Free Streaming for *iid* Initial Data

In the special case that $f_0^+ = f_0^-$ (which was our original assumption on the initial data), we see that our coupled Vlasov system simplifies to the free-streaming PDE (4.66). Hence, if we choose initial data *iid* by a single f_0, we get free-streaming in the infinitely many particles limit for all time $t \geq 0$. For completeness, we note that (4.66) has the obvious solution

$$f_t(p, q) = f_0(p, q - tv(p)).$$

In point of fact, if we examine the vector field associated to the free streaming PDE (4.66):

$$V^{fs}(t, p, q) = \begin{bmatrix} 0 \\ v(p) \end{bmatrix}, \tag{4.88}$$

we see that it has Lipschitz constant equal to 1. Hence, the associated flow $T_{t,0}^{fs}(p, q)$ has Lipschitz constant e^t. This allows us to replace $L_\pm$ appearing in the exponential term of (4.87) by 1 - yielding a slightly better rate of convergence in this special case.

4.8 The Vlasov Limit: *A Priori* Space-Time Scale

Our derivation of the traditional two-specie rVP system in §4.7.3 is not particularly unexpected given Neunzert's original single species derivation (as presented in §4.3). The necessary changes to pursue the convergence of two (or more) specie plasmas *on the traditional Vlasov scales* are fairly minimal.

However, on different space-time scales, most people have assumed that the collective behavior predicted by the Vlasov equations will be lost. Yet, the radical proposal presented in the appendix to [KTZ08] is that different collective effects could yet be seen on the *a priori* space-time scales!

In this section, we examine the behavior of our plasma model on the *a priori* space-time scales ($\alpha = 0$ in the notation of §4.6). Recall that on this scale, our point-particle dynamics is given by the PDE:

$$\partial_t {}^N\Delta_t^{\pm} + v(p) \cdot \nabla_q {}^N\Delta_t^{\pm} \pm NG_\epsilon * \left[{}^N\rho_t^+ - {}^N\rho_t^-\right](q) \cdot \nabla_p {}^N\Delta_t^{\pm} = 0. \tag{4.89}$$

Since we have assumed that the initial data for both particle species are chosen *iid* by f_0, we expect that the force field

$$NG_\epsilon * \left[{}^N\rho_t^+ - {}^N\rho_t^-\right](q)$$

will be close to zero for almost every point in space. However, for the finite number of points where charges reside, there will be an overall charge discrepancy since the self-interaction terms are all zero. Ferreting out the effect of this discrepancy for these finite number of points is delicate, as we will see shortly. In fact, while it is easy to show that initially the expected value of the force on each individual charge is precisely $-G_\epsilon * \rho_{f_0}(q)$, this is not enough to conclude that the force converges to this expression in probability.

So, we begin by setting up the basic notation and performing basic estimates (as in the previous sections). We will then examine how well rVP_ϵ^- approximates the particle dynamics. We shall see that for any *fixed* N, rVP_ϵ^- can provide a reasonable approximation for finite times. However, our estimates will not give us convergence in probability of the force term in the limit $N \to \infty$ even at the initial time! The last few sections will analyze this failure and indicate its likely resolution.

4.8.1 The Fixed Point Characterization and Lipschitz Estimates

As usual, we begin by associating vector fields on $\mathbb{R}^6$ to the PDEs (4.89) and (4.2).

$$V^{\pm}\left[{}^N\Delta_\cdot^+, {}^N\Delta_\cdot^-\right](t,p,q) = \begin{bmatrix} \pm NG_\epsilon * \left[{}^N\rho_t^+ - {}^N\rho_t^-\right](q) \\ v(p) \end{bmatrix}, \tag{4.90}$$

$$V^0[f_\cdot](t,p,q) = \begin{bmatrix} -G_\epsilon * \rho_{f_t}(q) \\ v(p) \end{bmatrix}. \tag{4.91}$$

called the *positive, negative, and neutral vector fields* respectively.

Associated to these vector fields are the corresponding flows $T^{\pm}_{t,0}[\cdot,\cdot]$ and $T^0_{t,0}[\cdot]$ (unsurprisingly called the positive, negative, and neutral flows). These are characterized by the following ODEs:

$$\begin{aligned} T^{\pm}_{t,0}\left[{}^N\!\Delta^+_\cdot, {}^N\!\Delta^-_\cdot\right](p,q) &= (p(t),q(t)) \\ \begin{bmatrix} \dot{p}(t) \\ \dot{q}(t) \end{bmatrix} &= V^{\pm}\left[{}^N\!\Delta^+_\cdot, {}^N\!\Delta^-_\cdot\right](t,p(t),q(t)) \\ (p(0),q(0)) &= (p,q). \end{aligned} \tag{4.92}$$

Similarly for $T^0[\cdot]$,

$$\begin{aligned} T^0_{t,0}[f_\cdot](p,q) &= (p(t),q(t)) \\ \begin{bmatrix} \dot{p}(t) \\ \dot{q}(t) \end{bmatrix} &= V^0[f_\cdot](t,p(t),q(t)) \\ (p(0),q(0)) &= (p,q). \end{aligned} \tag{4.93}$$

Note that the neutral vector fields and flows are exactly those given in §4.2.

As with all Hamiltonian flows, these flows preserve phase space volume in $\mathbb{R}^6$. Having established the relevant definitions, we can now see that our PDEs (4.89) and (4.2) are equivalent to:

$$ {}^N\!\Delta^{\pm}_t(p,q) = {}^N\!\Delta^{\pm}_0 \circ T^{\pm}_{0,t}\left[{}^N\!\Delta^+_\cdot, {}^N\!\Delta^-_\cdot\right](p,q), \tag{4.94}$$

$$ f_t(p,q) = f_0 \circ T^0_{0,t}[f_\cdot](p,q), \tag{4.95}$$

where $T^{\pm}_{0,t}, T^0_{0,t}$ are the backward flows on $\mathbb{R}^6$ (taking (p,q) to the initial condition $(\mathring{p},\mathring{q})$ where $T^{\cdot}_{t,0}(\mathring{p},\mathring{q}) = (p,q)$). Simply put, the flows $T^{\pm}_{0,t}$ and $T^0_{0,t}$ are associated to the vector fields $-V^{\pm}$ and $-V^0$ respectively.

We first establish some basic inequalities for our vector fields and flows. First, we establish Lipschitz constants for the neutral vector field and flow (again, this is a reiteration of the material found in §4.2). First, we have a t-independent Lipschitz constant for $V^0[f_\cdot]$:

$$\left|V^0[f_\cdot](t,p_1,q_1) - V^0[f_\cdot](t,p_2,q_2)\right| \le L_0 |(p_1,q_1)-(p_2,q_2)|, \tag{4.96}$$

where

$$L_0 = \max\{1, L_G\}. \tag{4.97}$$

and L_G was given by (4.10).

By a standard application of Gronwall's Inequality (c.f. [Ro95] p.142), we have a Lipschitz constant for $T^0_{t,0}[f_\cdot]$:

$$\left|T^0_{t,0}[f_\cdot](p_1,q_1) - T^0_{t,0}[f_\cdot](p_2,q_2)\right| \le e^{L_0 t}|(p_1,q_1)-(p_2,q_2)|. \tag{4.98}$$

We also note that both V^0 and $T^0_{t,0}$ are continuous as functions of t.

There are corresponding estimates for $V^\pm$ and $T^\pm$, but unfortunately, these depend upon N.

$$\begin{aligned} N\left|\int \left(G_\epsilon(q_1-q') \right.\right. & \left.\left. -G_\epsilon(q_2-q')\right)\left[{}^N\!\rho^+_t(q') - {}^N\!\rho^-_t(q')\right] d^3q' \right| \\ & \le N L_G|q_1-q_2| \int {}^N\!\rho^+_t(q') + {}^N\!\rho^-_t(q') d^3q' \\ & \le 2NL_G|q_1-q_2| \end{aligned}$$

Hence, we get a Lipschitz constant for the charged vector fields and flows:

$$\left|V^\pm\left[{}^N\!\Delta^+_\cdot, {}^N\!\Delta^-_\cdot\right](t,p_1,q_1) - V^\pm\left[{}^N\!\Delta^+_\cdot, {}^N\!\Delta^-_\cdot\right](t,p_2,q_2)\right| \le L_\pm|(p_1,q_1)-(p_2,q_2)|, \tag{4.99}$$

and

$$\left|T^\pm_{t,0}\left[{}^N\!\Delta^+_\cdot, {}^N\!\Delta^-_\cdot\right](p_1,q_1) - T^\pm_{t,0}\left[{}^N\!\Delta^+_\cdot, {}^N\!\Delta^-_\cdot\right](p_2,q_2)\right| \le e^{L_\pm t}|(p_1,q_1)-(p_2,q_2)|, \tag{4.100}$$

where

$$L_\pm = \max\{1, 2NL_G\}. \tag{4.101}$$

4.8.2 Approximation by $\mathbf{rVP}^-_\epsilon$

We are now in a position to begin the proof that rVP^-_ϵ approximates our particle dynamics introduced in (4.33) - (4.36). To be concrete, we will consider ${}^N\!\Delta^+_t$; charge symmetry will give analogous results for ${}^N\!\Delta^-_t$. In what follows, φ will denote an arbitrary element of $\mathcal{D}(\mathbb{R}^6)$ as defined in (4.25).

$$\begin{aligned}
&\left|\iint \varphi(p,q)\left({}^N\Delta_t^+(p,q) - f_t(p,q)\right) d^3p d^3q\right| \\
&\quad = \left|\iint \varphi(p,q)\left({}^N\Delta_0^+ \circ T_{0,t}^+\left[{}^N\Delta_\cdot^+, {}^N\Delta_\cdot^-\right](p,q) - f_0 \circ T_{0,t}^0[f_\cdot](p,q)\right) d^3p d^3q\right| \\
&\quad \leq \left|\iint \varphi(p,q)\left({}^N\Delta_0^+ \circ T_{0,t}^+\left[{}^N\Delta_\cdot^+, {}^N\Delta_\cdot^-\right](p,q) - {}^N\Delta_0^+ \circ T_{0,t}^0[f_\cdot](p,q)\right) d^3p d^3q\right| \\
&\qquad + \left|\iint \varphi(p,q)\left({}^N\Delta_0^+ \circ T_{0,t}^0[f_\cdot](p,q) - f_0 \circ T_{0,t}^0[f_\cdot](p,q)\right) d^3p d^3q\right| \\
&\quad \leq \iint \left|\varphi \circ T_{t,0}^+\left[{}^N\Delta_\cdot^+, {}^N\Delta_\cdot^-\right](p,q) - \varphi \circ T_{t,0}^0[f_\cdot](p,q)\right| {}^N\Delta_0^+(p,q) d^3p d^3q \\
&\qquad + \left|\iint \varphi \circ T_{t,0}^0[f_\cdot](p,q)\left({}^N\Delta_0^+(p,q) - f_0(p,q)\right) d^3p d^3q\right|
\end{aligned}$$

For the second term in the last inequality, we note that (4.98) implies $e^{-L_0 t}\varphi \circ T_{t,0}^0[f_\cdot] \in \mathcal{D}(\mathbb{R}^6)$. Hence, we get the inequality

$$\begin{aligned}
d_{bL^*}({}^N\Delta_t^+, f_t) \leq{} & e^{L_0 t} d_{bL^*}({}^N\Delta_0^+, f_0) \\
& + \iint \left|T_{t,0}^+\left[{}^N\Delta_\cdot^+, {}^N\Delta_\cdot^-\right](p,q) - T_{t,0}^0[f_\cdot](p,q)\right| {}^N\Delta_0^+(p,q) d^3p d^3q. \qquad (4.102)
\end{aligned}$$

Since the first term is exactly the sort of bound we would like, we need only focus on the second term. We make the following definitions:

$$\lambda^+(t) = \iint \left|T_{t,0}^+\left[{}^N\Delta_\cdot^+, {}^N\Delta_\cdot^-\right](p,q) - T_{t,0}^0[f_\cdot](p,q)\right| {}^N\Delta_0^+(p,q) d^3p d^3q, \qquad (4.103)$$

$$\lambda^-(t) = \iint \left|T_{t,0}^-\left[{}^N\Delta_\cdot^+, {}^N\Delta_\cdot^-\right](p,q) - T_{t,0}^0[f_\cdot](p,q)\right| {}^N\Delta_0^-(p,q) d^3p d^3q. \qquad (4.104)$$

With these functions, we can write

$$d_{bL^*}({}^N\Delta_t^+, f_t) + d_{bL^*}({}^N\Delta_t^-, f_t) \leq e^{L_0 t}\left(d_{bL^*}({}^N\Delta_0^+, f_0) + d_{bL^*}({}^N\Delta_0^-, f_0)\right) + \lambda^+(t) + \lambda^-(t). \qquad (4.105)$$

Iterating the flow gives

$$
\begin{aligned}
&\Big| T^{+}_{t,0}\left[{}^N\!\Delta^{+}_{\cdot}, {}^N\!\Delta^{-}_{\cdot}\right](p,q) - T^{0}_{t,0}[f_{\cdot}](p,q)\Big| \\
&\leq \int_0^t \Big| V^{+}\left[{}^N\!\Delta^{+}_{\cdot}, {}^N\!\Delta^{-}_{\cdot}\right]\Big(\tau, T^{+}_{\tau,0}\left[{}^N\!\Delta^{+}_{\cdot}, {}^N\!\Delta^{-}_{\cdot}\right](p,q)\Big) - V^{0}[f_{\cdot}]\left(\tau, T^{0}_{\tau,0}[f_{\cdot}](p,q)\right)\Big| d\tau \\
&\leq \int_0^t \Big| V^{+}\left[{}^N\!\Delta^{+}_{\cdot}, {}^N\!\Delta^{-}_{\cdot}\right]\Big(\tau, T^{+}_{\tau,0}\left[{}^N\!\Delta^{+}_{\cdot}, {}^N\!\Delta^{-}_{\cdot}\right](p,q)\Big) - V^{0}[f_{\cdot}]\Big(\tau, T^{+}_{\tau,0}\left[{}^N\!\Delta^{+}_{\cdot}, {}^N\!\Delta^{-}_{\cdot}\right](p,q)\Big)\Big| d\tau \\
&\quad + \int_0^t \Big| V^{0}[f_{\cdot}]\Big(\tau, T^{+}_{\tau,0}\left[{}^N\!\Delta^{+}_{\cdot}, {}^N\!\Delta^{-}_{\cdot}\right](p,q)\Big) - V^{0}[f_{\cdot}]\left(\tau, T^{0}_{\tau,0}[f_{\cdot}](p,q)\right)\Big| d\tau \\
&\leq \int_0^t \Big| V^{+}\left[{}^N\!\Delta^{+}_{\cdot}, {}^N\!\Delta^{-}_{\cdot}\right]\Big(\tau, T^{+}_{\tau,0}\left[{}^N\!\Delta^{+}_{\cdot}, {}^N\!\Delta^{-}_{\cdot}\right](p,q)\Big) - V^{0}[f_{\cdot}]\Big(\tau, T^{+}_{\tau,0}\left[{}^N\!\Delta^{+}_{\cdot}, {}^N\!\Delta^{-}_{\cdot}\right](p,q)\Big)\Big| d\tau \\
&\quad + L_0 \int_0^t \Big| T^{+}_{\tau,0}\left[{}^N\!\Delta^{+}_{\cdot}, {}^N\!\Delta^{-}_{\cdot}\right](p,q) - T^{0}_{\tau,0}[f_{\cdot}](p,q)\Big| d\tau
\end{aligned}
$$

where the latter half of the last inequality follows from (4.12).

Integrating both sides of the last inequality against ${}^N\!\Delta^{+}_{0}(p,q)d^3pd^3q$ and defining

$$
\gamma^{+}(t) = \iint \left| V^{+}\left[{}^N\!\Delta^{+}_{\cdot}, {}^N\!\Delta^{-}_{\cdot}\right](t,p,q) - V^{0}[f_{\cdot}](t,p,q)\right| {}^N\!\Delta^{+}_{t}(p,q)d^3pd^3q \tag{4.106}
$$

(and similarly γ^{-}) gives the inequality

$$
\lambda^{+}(t) \leq L_0 \int_0^t \lambda^{+}(\tau)d\tau + \int_0^t \gamma^{+}(\tau)d\tau,
$$

and a simple corollary to Gronwall's Inequality gives

$$
\lambda^{+}(t) \leq e^{L_0 t} \int_0^t \gamma^{+}(\tau)e^{-L_0\tau}d\tau. \tag{4.107}
$$

We now focus on estimating $\gamma^{+}(t)$. Tracing back through the various definitions gives

$$
\begin{aligned}
\gamma^{+}(t) &= \iint \left| G_\epsilon * \left[N\,{}^N\!\rho^{+}_{t} - N\,{}^N\!\rho^{-}_{t} + \rho_{f_t}\right](q)\right| {}^N\!\Delta^{+}_{t}(p,q)d^3pd^3q \\
&= \sum_{i=1}^{N} \left| G_\epsilon * \left[{}^N\!\rho^{+}_{t} - {}^N\!\rho^{-}_{t} + \frac{1}{N}\rho_{f_t}\right](q_{2i}(t))\right|.
\end{aligned}
$$

Looking carefully at the definition of G_ϵ shows that

$$
G_\epsilon(0) = 0
$$

(which is nothing more than our assumption that self-interactions are zero). Hence, we can ignore the pairing of particle $2i$ with itself in the above equality. With that in mind, we make the following definitions:

$$ {}^{N\backslash j}\Delta_0^+(p,q) = \frac{1}{N-1}\sum_{i\neq j}\delta(p-p_{2i}(0))\delta(q-q_{2i}(0)), \quad (4.108) $$

$$ {}^{N\backslash j}\Delta_0^-(p,q) = \frac{1}{N-1}\sum_{i\neq j}\delta(p-p_{2i-1}(0))\delta(q-q_{2i-1}(0)). \quad (4.109) $$

Note that each of these finite measures has total integral 1.

At later times, we define

$$ {}^{N\backslash j}\Delta_t^\pm(p,q) = {}^{N\backslash j}\Delta_0^\pm \circ T_{0,t}^\pm\left[{}^N\Delta_\cdot^+, {}^N\Delta_\cdot^-\right](p,q). \quad (4.110) $$

Thus, ${}^{N\backslash j}\Delta_t^\pm$ tracks the motion of all but the j-th particle in that species, BUT the motion is still determined by all $2N$ particles. We also make the analogous definitions for ${}^{N\backslash j}\rho_t^\pm$.

Now we have

$$ \begin{aligned} \gamma^+(t) &= \sum_{i=1}^N \left| G_\epsilon * \left[\frac{N-1}{N}{}^{N\backslash i}\rho_t^+ - {}^N\rho_t^- + \frac{1}{N}\rho_{f_t}\right](q_{2i}(t))\right| \\ &\leq \frac{N-1}{N^2}\sum_{i=1}^N\sum_{j=1}^N \left| G_\epsilon * \left[{}^{N\backslash i}\rho_t^+ - {}^{N\backslash j}\rho_t^-\right](q_{2i}(t))\right| \\ &\quad + \frac{1}{N}\sum_{i=1}^N \left| G_\epsilon * \left[\rho_{f_t} - {}^N\rho_t^-\right](q_{2i}(t))\right| \end{aligned} $$

where in the last term, we have averaged over the removal of the j-th negative particle.

Now, each component of G_ϵ is Lipschitz continuous with Lipschitz constant given by (4.10). So,

$$ \gamma^+(t) \leq \sqrt{3}L_G\frac{N-1}{N^2}\sum_{i=1}^N\sum_{j=1}^N d_{bL^*}\left({}^{N\backslash i}\Delta_t^+, {}^{N\backslash j}\Delta_t^-\right) + \sqrt{3}L_G d_{bL^*}\left({}^N\Delta_t^-, f_t\right), $$

where the $\sqrt{3}$ accounts for the fact that the estimate covers three components of the vector-valued function G_ϵ.

We now make the obvious estimate:

$$ d_{bL^*}\left({}^{N\backslash i}\Delta_t^+, {}^{N\backslash j}\Delta_t^-\right) \leq d_{bL^*}\left({}^{N\backslash i}\Delta_t^+, f_t\right) + d_{bL^*}\left({}^{N\backslash j}\Delta_t^-, f_t\right) $$

which gives

$$ \gamma^+(t) \leq \sqrt{3}L_G\frac{N-1}{N}\sum_{i=1}^N\left(d_{bL^*}\left({}^{N\backslash i}\Delta_t^+, f_t\right) + d_{bL^*}\left({}^{N\backslash i}\Delta_t^-, f_t\right)\right) + \sqrt{3}L_G d_{bL^*}\left({}^N\Delta_t^-, f_t\right). $$

Combining this with (4.107) and the analogous results by charge symmetry gives

$$
\begin{aligned}
&e^{-L_0 t}\left(\lambda^+(t)+\lambda^-(t)\right)\\
&\qquad\le 2\sqrt{3}L_G\frac{N-1}{N}\sum_{i=1}^{N}\int_0^t\left(d_{bL^*}\left({}^{N\setminus i}\Delta_\tau^+,f_\tau\right)+d_{bL^*}\left({}^{N\setminus i}\Delta_t^-,f_\tau\right)\right)e^{-L_0\tau}d\tau\\
&\qquad\quad+\sqrt{3}L_G\int_0^t\left(d_{bL^*}\left({}^{N}\Delta_\tau^+,f_\tau\right)+d_{bL^*}\left({}^{N}\Delta_\tau^-,f_\tau\right)\right)e^{-L_0\tau}d\tau.
\end{aligned}
$$

Combining this with (4.105) gives

$$
\begin{aligned}
&e^{-L_0 t}\left(d_{bL^*}({}^{N}\Delta_t^+,f_t)+d_{bL^*}({}^{N}\Delta_t^-,f_t)\right)\\
&\qquad\le d_{bL^*}({}^{N}\Delta_0^+,f_0)+d_{bL^*}({}^{N}\Delta_0^-,f_0)\\
&\qquad\quad+2\sqrt{3}L_G\frac{N-1}{N}\sum_{i=1}^{N}\int_0^t\left(d_{bL^*}\left({}^{N\setminus i}\Delta_\tau^+,f_\tau\right)+d_{bL^*}\left({}^{N\setminus i}\Delta_t^-,f_\tau\right)\right)e^{-L_0\tau}d\tau\\
&\qquad\quad+\sqrt{3}L_G\int_0^t\left(d_{bL^*}\left({}^{N}\Delta_\tau^+,f_\tau\right)+d_{bL^*}\left({}^{N}\Delta_\tau^-,f_\tau\right)\right)e^{-L_0\tau}d\tau.
\end{aligned}
$$

Yet another application of Gronwall's Inequality gives the very useful estimate

$$
\begin{aligned}
&e^{-(\sqrt{3}L_G+L_0)t}\left(d_{bL^*}({}^{N}\Delta_t^+,f_t)+d_{bL^*}({}^{N}\Delta_t^-,f_t)\right)\\
&\quad\le d_{bL^*}({}^{N}\Delta_0^+,f_0)+d_{bL^*}({}^{N}\Delta_0^-,f_0)\\
&\qquad+2\sqrt{3}L_G\frac{N-1}{N}\sum_{i=1}^{N}\int_0^t\left(d_{bL^*}\left({}^{N\setminus i}\Delta_\tau^+,f_\tau\right)+d_{bL^*}\left({}^{N\setminus i}\Delta_\tau^-,f_\tau\right)\right)e^{-(\sqrt{3}L_G+L_0)\tau}d\tau
\end{aligned}
\tag{4.111}
$$

Unfortunately, these estimates do not allow us to prove *convergence in probability* for any time beyond $t=0$. So, we look at expectation values. From (4.111), the fact that our labeling is arbitrary, and that for large N we can ignore the difference between having N particles and $N-1$ particles we see

$$
\begin{aligned}
&e^{-(\sqrt{3}L_G+L_0)t}\mathbb{E}_{\mathbb{P}_0}\left[d_{bL^*}({}^{N}\Delta_t^+,f_t)+d_{bL^*}({}^{N}\Delta_t^-,f_t)\right]\\
&\qquad\lesssim\mathbb{E}_{\mathbb{P}_0}\left[d_{bL^*}({}^{N}\Delta_0^+,f_0)+d_{bL^*}({}^{N}\Delta_0^-,f_0)\right]\\
&\qquad\quad+2\sqrt{3}L_G(N-1)\int_0^t\mathbb{E}_{\mathbb{P}_0}\left[d_{bL^*}\left({}^{N}\Delta_\tau^+,f_\tau\right)+d_{bL^*}\left({}^{N}\Delta_\tau^-,f_\tau\right)\right]e^{-(\sqrt{3}L_G+L_0)\tau}d\tau.
\end{aligned}
\tag{4.112}
$$

A final application of Gronwall's Inequality and the fact that $L_G \leq L_0$ (by definition) gives

$$\mathbb{E}_{\mathbb{P}_0}\left[d_{bL^*}({}^N\!\Delta_t^+, f_t) + d_{bL^*}({}^N\!\Delta_t^-, f_t)\right] \lesssim e^{2\sqrt{3}L_0 N t}\mathbb{E}_{\mathbb{P}_0}\left[d_{bL^*}({}^N\!\Delta_0^+, f_0) + d_{bL^*}({}^N\!\Delta_0^-, f_0)\right] \tag{4.113}$$

Our results from §4.5 tell us that if $f_0 d^3p d^3q$ satisfies LSI(κ, λ) then for large N the expectation value of

$$d_{bL^*}({}^N\!\Delta_0^+, f_0) + d_{bL^*}({}^N\!\Delta_0^-, f_0)$$

is bounded above by $CN^{-2/1+\kappa}$ where C depends on λ and κ but is independent of N. Hence (4.113) gives us that

$$\mathbb{E}_{\mathbb{P}_0}\left[d_{bL^*}({}^N\!\Delta_t^+, f_t) + d_{bL^*}({}^N\!\Delta_t^-, f_t)\right] \lesssim \frac{Ce^{2\sqrt{3}L_0 N t}}{N^{\frac{2}{1+\kappa}}}. \tag{4.114}$$

Thus, we get convergence in expectation *on time-scales of order 1/N*.

Our findings are not dissimilar to the state of affairs for the Boltzmann Equation. Lanford's derivation of the Boltzmann Equation (see [La83] for a nice summary of the results) is valid for times of order $(Nd^2)^{-1}$ where N is the number of particles and d is their diameter. Of course, the regime in which the infinite particle limit is obtained is the so-called Boltzmann-Grad limit where d tends to zero as N goes to infinity in such a way that Nd^2 limits to a finite value.

Looking at (4.112), we see that we could get convergence at all later times by requiring $L_G N$ to tend to some finite quantity as N tends to infinity (we replaced L_G by L_0 simply to make (4.114) look more pleasant). Looking at the definition of L_G given in (4.10), we see that this forces $N\epsilon^{-3}$ to limit to something finite (recall that the parameter ϵ essentially gives the radius of our particles). Hence, we would need to require that the our particles grow in radius on the order of $N^{1/3}$. From a physical point of view, this is precisely the opposite of what we would like to have occur!

For *fixed* N we can have that rVP$_\epsilon^-$ is a reasonable approximation to the underlying dynamics for much longer times than $1/N$. To see this, note that nothing in our derivation depends on the size of supp(f_0). If we assume the effective radius of the support of f_0 in q-space is large compared to N (some power of N large compared to

1/3 will do), then assuming that the radius of the particles is on the order of $N^{1/3}$ will not be so ridiculous. The details, of course, will certainly depend on the particular system of units employed.

4.8.3 Examination of the Force Term at the Initial Time

In the previous section, we have shown that rVP_ϵ^- approximates our particle dynamics reasonably well for times up to order $1/N$ (or potentially longer times for any fixed N). A natural question to ponder is whether we can do better than the estimates in the previous section. To shed some light on this, we look closely at the expected force felt by any one of the particles at the initial time. To make things definite, consider one of the negative particles (by our *iid* assumption and charge symmetry, the results are easily adapted to any of the particles in our plasma). We can imagine that at time zero, we place N positive particles *iid* by $f_0(p,q)d^3pd^3q$ and $N-1$ negative particles chosen *iid* in the same manner. Pick any point q in the support of f_0. The expected value of the force produced by the distribution at the point q will be the expected value of the force felt by the remaining negative particle (again, thanks to our regularization). Hence, the expected value of

$$-G_\epsilon * \left[N\,{}^{N}\!\rho_0^{+} - (N-1)\,{}^{N-1}\!\rho_0^{-}\right](q), \tag{4.115}$$

is the expected force felt by any one of the negative particles. To be definite, we assume the missing negative particle is to be labeled by "1", and so we have already placed particles "2" through "$2N$".

$$\begin{aligned}
&\mathbb{E}_{\mathbb{P}_0}\left[-G_\epsilon * \left[N\,{}^N\!\rho_0^+ - (N-1)\,{}^{N-1}\!\rho_0^-\right](q)\right] \\
&\quad = -\int_{\mathbb{R}^{12N-6}} G_\epsilon * \left[N\,{}^N\!\rho_0^+ - (N-1)\,{}^{N-1}\!\rho_0^-\right](q) \bigotimes_{i=2}^{2N} f_0 \; d^3p\, d^3q \\
&\quad = -\int_{\mathbb{R}^{12N-6}} \left[\sum_{i=1}^{N} G_\epsilon(q, q_{2i}) - \sum_{j=2}^{N} G_\epsilon(q, q_{2j-1})\right] \bigotimes_{i=2}^{2N} f_0 \; d^3p\, d^3q \\
&\quad = -\sum_{i=1}^{N} \iint G_\epsilon(q, q_{2i}) f_0(p_{2i}, q_{2i}) \; d^3p_{2i}\, d^3q_{2i} \\
&\qquad\qquad + \sum_{j=2}^{N} \iint G_\epsilon(q, q_{2j-1}) f_0(p_{2j-1}, q_{2j-1}) \; d^3p_{2j-1}\, d^3q_{2j-1} \\
&\quad = -\iint G_\epsilon(q, q') f_0(p', q') \; d^3p'\, d^3q' \\
&\quad = -G_\epsilon * \rho_{f_0}(q)
\end{aligned}$$

Thus,

$$\mathbb{E}_{\mathbb{P}_0}\left[-G_\epsilon * \left[N\,{}^N\!\rho_0^+ - (N-1)\,{}^{N-1}\!\rho_0^-\right](q)\right] \quad = \quad -G_\epsilon * \rho_{f_0}(q) \tag{4.116}$$

for any position q and for any N. By charge symmetry, we immediately have that the force felt by any one of the positive charges at the initial time also has expectation

$$\mathbb{E}_{\mathbb{P}_0}\left[G_\epsilon * \left[(N-1)\,{}^{N-1}\!\rho_0^+ - N\,{}^N\!\rho_0^-\right](q)\right] \quad = \quad -G_\epsilon * \rho_{f_0}(q). \tag{4.117}$$

The net result of this is that *at the initial time* we get our regularized version of rVP$^-$ *in expected value*. Indeed, this was the primary motivation in [KTZ08] for considering the plasma model in the first place.†

In our previous considerations, we were asking for stronger information about the convergence. The expectation above is an *ensemble* average, while in the previous section we were asking for expected rates of convergence for any given realization of the ensemble. To see the difference, note that by the above calculation we expect the random empirical measures ${}^N\!\rho_0^+$ and ${}^{N-1}\!\rho_0^-$ to be equal *when averaged over the*

---

†Of course, the fact that the expected value of the force equals the mean-field force at the initial time does not imply that this is true at subsequent times. This is easily seen by for instance by discussing an ensemble of two-body systems. See the appendix to this chapter - §4.11.

entire ensemble. Of course, any particular realization of the system may have extreme charge separation, and so the force on a given particle may depart significantly from $-G_\epsilon * \rho_{f_0}(q)$. By choosing all of the particles *iid*, we essentially force cancelation by ensuring any particular realization of the system with significant charge separation is exactly balanced by a companion case with the opposite separation. To examine what this means for our convergence in previous sections, we need to consider the expected rate of convergence of the force to $-G_\epsilon * \rho_{f_0}(q)$.

To that end, we consider

$$\mathbb{E}_{\mathbb{P}_0}\left[\left|G_\epsilon * \left[-\rho_{f_0} + N\,{}^{N}\!\rho_0^{+} - (N-1)\,{}^{N-1}\!\rho_0^{-}\right](q)\right|\right], \tag{4.118}$$

and attempt to estimate its size in terms of the dual, bounded Lipschitz distance. We make the obvious estimate

$$\begin{aligned} |G_\epsilon *& \left[-\rho_{f_0} + N\,{}^{N}\!\rho_0^{+} - (N-1)\,{}^{N-1}\!\rho_0^{-}\right](q)| \\ &\leq |G_\epsilon * \left[{}^{N}\!\rho_0^{+} - \rho_{f_0}\right](q)| + (N-1)|G_\epsilon * \left[{}^{N}\!\rho_0^{+} - {}^{N-1}\!\rho_0^{-}\right](q)| \end{aligned} \tag{4.119}$$

Note that the two terms on the RHS above are essentially independent for any given realization of the system. Of course, they are not independent random variables over our ensemble since a system where the first term above is large will most likely have significant charge separation.

We will use the contraction principle to estimate the probability that each of the terms appearing is larger than some δ. Integrating this will give an estimate on the expected value. The first term on the RHS of (4.119) satisfies a large deviation principle given by

$$\begin{aligned} \mathbb{P}_0\left(|G_\epsilon * \left[{}^{N}\!\rho_0^{+} - \rho_{f_0}\right](q)| > \delta\right) &\asymp e^{-\mathcal{H}_1(\delta)N}, \\ \inf\left\{H(\mu|\rho_{f_0}) \,:\, |G_\epsilon * \left[\mu - \rho_{f_0}\right](q)| > \delta\right\} &= \mathcal{H}_1(\delta) \end{aligned}$$

where the infimum is taken over all probability measures on $\mathbb{R}^3$ satisfying the stated condition. Since the components of G_ϵ are Lipschitz continuous and bounded (by L_G), the condition

$$|G_\epsilon * \left[\mu - \rho_{f_0}\right](q)| > \delta$$

implies that

$$d_{bL^*}(\mu, \rho_{f_0}) > \frac{\delta}{\sqrt{3}L_G}$$

(the $\sqrt{3}$ coming from the fact that G_ϵ has three components). If f_0 satisfies LSI(κ, λ), then we have

$$H(\mu|\rho_{f_0}) > \frac{\lambda}{1+\kappa}\left(\frac{\delta}{\sqrt{3}L_G}\right)^{1+\kappa}.$$

Integrating this to get the expectation value gives

$$\mathbb{E}_{\mathbb{P}_0}\left[\left|G_\epsilon * \left[{}^N\!\rho_0^+ - \rho_{f_0}\right](q)\right|\right] \lesssim \frac{\Gamma\left(\frac{2}{1+\kappa}\right)(1+\kappa)^{\frac{1-\kappa}{1+\kappa}}(\sqrt{3}L_G)^2}{(\lambda N)^{\frac{2}{1+\kappa}}}. \tag{4.120}$$

For the second term, we need a version of the contraction principle that applies to a continuous function of two variables. Such a principle is readily found (c.f. Exercise 4.2.7 in [DZ98]). Namely, we have

$$\begin{aligned} \mathbb{P}_0\left(\left|G_\epsilon * \left[{}^N\!\rho_0^+ - {}^{N-1}\!\rho_0^-\right](q)\right| > \delta\right) &\asymp e^{-\mathcal{H}_2(\delta)N} \\ \inf\left\{H(\mu|\rho_{f_0}) + H(\nu|\rho_{f_0}) \;:\; \left|G_\epsilon * [\mu - \nu](q)\right| > \delta\right\} &= \mathcal{H}_2(\delta) \end{aligned}$$

where the infimum is taken over all pairs (μ, ν) of probability measures on $\mathbb{R}^3$ satisfying the stated condition (this principle is akin to the additivity of entropy in statistical physics). The same reasoning as before shows that

$$d_{bL^*}(\mu, \rho_{f_0}) + d_{bL^*}(\nu, \rho_{f_0}) \geq d_{bL^*}(\mu, \nu) > \frac{\delta}{\sqrt{3}L_G},$$

and so, one of the two summands on the left above must be larger than half the lower bound. Without loss of generality, assume

$$d_{bL^*}(\mu, \rho_{f_0}) > \frac{\delta}{2\sqrt{3}L_G}.$$

Our assumption that f_0 satisfies LSI(κ, λ) gives us that

$$\mathcal{H}_2(\delta) > \frac{\lambda}{1+\kappa}\left(\frac{\delta}{2\sqrt{3}L_G}\right)^{1+\kappa}.$$

Integrating this gives

$$\mathbb{E}_{\mathbb{P}_0}\left[\left|G_\epsilon * \left[{}^N\!\rho_0^+ - {}^{N-1}\!\rho_0^-\right](q)\right|\right] \lesssim \frac{4\Gamma\left(\frac{2}{1+\kappa}\right)(1+\kappa)^{\frac{1-\kappa}{1+\kappa}}(\sqrt{3}L_G)^2}{(\lambda N)^{\frac{2}{1+\kappa}}}. \tag{4.121}$$

Recall that we need to multiply this by a factor of $N-1$ in (4.119).

Thus, if f_0 satisfies LSI(κ,λ) we expect

$$\mathbb{E}_{\mathbb{P}_0}\left[\left|G_\epsilon * \left[-\rho_{f_0} + N\,{}^N\!\rho_0^+ - (N-1)\,{}^{N-1}\!\rho_0^-\right](q)\right|\right] \asymp \frac{\mathcal{C}}{N^{\frac{1-\kappa}{1+\kappa}}} \tag{4.122}$$

where $\mathcal{C}$ depends on λ,κ,ϵ and the regularizer η. As we demonstrated in §4.5.2, the only admissible exponents κ are $\kappa \geq 1$. But this means we have no control over the convergence in expectation of the discrepancy of the force term from its expected value!

Similarly, the variance

$$\mathrm{Var}_{\mathbb{P}_0}\left(\left|G_\epsilon * \left[-\rho_{f_0} + N\,{}^N\!\rho_0^+ - (N-1)\,{}^{N-1}\!\rho_0^-\right](q)\right|\right),$$

will be controlled by

$$(N-1)^2\mathbb{E}_{\mathbb{P}_0}\left[\left|G_\epsilon * \left[{}^N\!\rho_0^+ - {}^{N-1}\!\rho_0^-\right](q)\right|^2\right].$$

Applying the same reasoning shows that the variance will behave as

$$\mathrm{Var}_{\mathbb{P}_0}\left(\left|G_\epsilon * \left[-\rho_{f_0} + N\,{}^N\!\rho_0^+ - (N-1)\,{}^{N-1}\!\rho_0^-\right](q)\right|\right) \asymp \frac{\mathcal{C}}{N^{\frac{2-2\kappa}{1+\kappa}}}. \tag{4.123}$$

4.9 The Distribution of Particles After the Initial Time

Since by (4.114) rVP_ϵ^- approximates the N-body dynamics for sufficiently large but fixed number of particles N up to some finite time (say $t_0(N)$), a natural question to ask is whether we can bootstrap these results to apply to times after $t_0(N)$. That is, can we restart the process used in the proof taking the data at t_0 as new initial data? If so, can we continue this bootstrapping ad infinitum? If this were the case, then the dynamics would remain close to that of rVP_ϵ^- for all times for any N (sufficiently large). Unfortunately, for finite N the answer has to be 'No!' in general. The reason is that we can no longer be certain that our ensemble consists of particles chosen *independently* after t_0; that is, for finite N-body systems we expect the build-up of correlations after the initial time (as before see the appendix — §4.11).

However, the rVP^- force term will retain its significance in the $N \to \infty$ limit (provided this exists), as we explain now. Recall that we used the *iid* assumption to give a rate of convergence for initial data via Sanov's Theorem - an essential part of the proofs above. We also used the fact that the particles are *identically* distributed when we averaged over the removal of certain particles. This is unchanged by the particle dynamics since said dynamics is symmetric in each species. Now, if the individual empirical 1-point densities at time t converge at all — to $f_t^{\pm}$, say, then the empirical n-point densities for each species at time t must converge to products of the limiting 1-point densities at time t. Moreover, charge symmetry gives us that these limiting 1-point densities at time t should be identical for both species (i.e. $f_t^+ = f_t^- = f_t$). Hence, in the $N \to \infty$ limit we recover that all the particles are distributed *iid* by f_t. Therefore, the expected value of the force remains $-G_\epsilon * \rho_{f_t}$, except that the density f_t may not be evolved from f_0 by rVP_ϵ^-.

There are three immediate scenarios which could explain our results. The first is that the evolution of f_t is really given by rVP_ϵ^-, but our upper bounds are simply not sharp enough to show the convergence. What seems most likely though is that there is a significant discrete particle effect contributing to the dynamics *in addition to* the collective effects of rVP_ϵ^-. This could be due to the influence of some "collision" operator (e.g. a relativistic version of the Lenard-Balescu operator). However, another possibility we see is that instead of a deterministic collision-type operator, there is some overall stochastic operator contributing to the dynamics of f_t.

We should note that if our expectations are borne out and there is some collision-type operator joining the rVP^- force term, then we can schematically write our alleged PDE for f_t as

$$\partial_t f_t + v(p) \cdot \nabla_q f_t - G_\epsilon * \rho_{f_t}(q) \cdot \nabla_p f_t = \mathfrak{C}(f_t), \tag{4.124}$$

where $\mathfrak{C}$ represents whatever "collision" operator is influencing the dynamics. Assuming that $\mathfrak{C}(f) = 0$ if and only if f is the Boltzmann-Jüttner distribution

$$f_J(p,q) = \mathcal{C} e^{-\beta\left(\sqrt{1+|p|^2}+\psi(q)\right)},$$

(which is the case for all the known operators we have in mind), then any stationary state of our PDE is given by f_J and will satisfy the *stationary* relativistic Vlasov-Poisson system:

$$v(p) \cdot \nabla_q f_J - G_\epsilon * \rho_{f_J}(q) \cdot \nabla_p f_J = 0.$$

Note that the p-derivative in the second term on the LHS will give us back $f_J(p,q)v(p)$ up to some constant. Thus, we are left with a PDE for ψ of the form

$$\nabla \psi(q) = -\mathcal{C} G_\epsilon * \rho_{f_J}(q).$$

Recalling that G_ϵ can be written as the gradient of the regularized Coulomb potential gives in the $\epsilon \to 0$ limit the well-known PDE of Emden's isothermal gas spheres [Em07]:

$$\Delta \psi(q) = \mathcal{C} e^{-\beta \psi}.$$

Of course in Emden's β, the coupling constant NGm^2 is now replaced by e^2, in the appropriate physical units. This PDE has been studied extensively in the astrophysics literature (e.g. [Ch67] pp. 155-182). The solutions to this PDE describe highly non-uniform densities in physical space. Therefore, this equation suggests the possible existence of spatially non-uniform spherical electrostatic plasma equilibria.

4.10 Summary and Future Directions

4.10.1 The Findings of this Chapter

To summarize our results, (4.114) shows that for any fixed N rVP$_\epsilon^-$ is a viable approximation to the underlying N-body dynamics of an overall neutral two-specie plasma for times at least of order $1/N$ on the *a priori* space-time scales. For a given system, the actual time over which rVP$_\epsilon^-$ remains a decent approximation on this scale depends on the relative size of the supports of our regularizer η_ϵ and the function f_0.

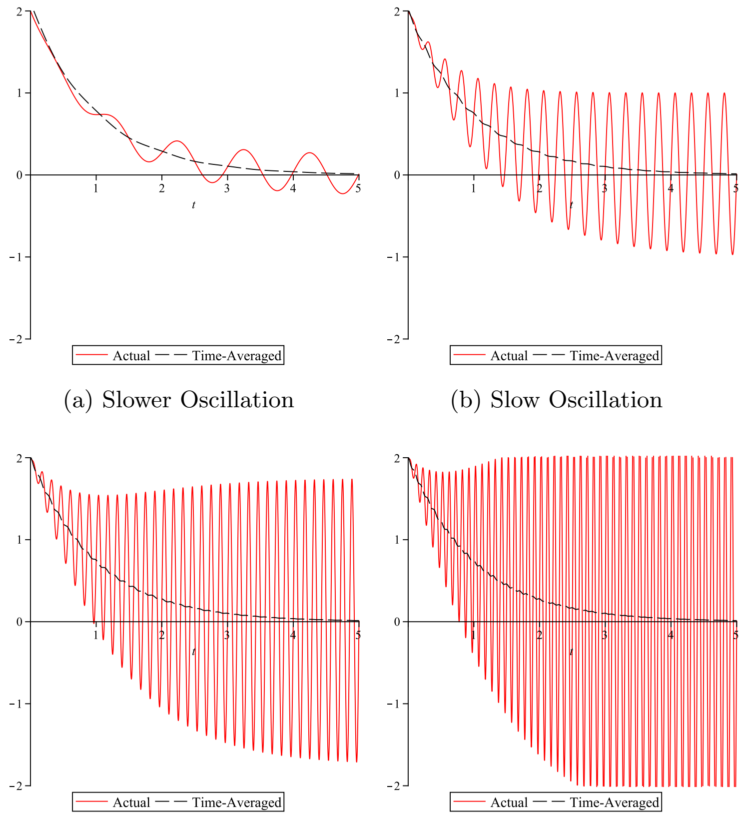
Moreover, we have established the essential roadblock to pushing these results from mere approximation to an actual continuum limit. Namely, the spatial charge separation ${}^N\rho_t^+ - {}^N\rho_t^-$ responsible for the force converges *at best* as $1/N$ on average (the best case occurring when f_0 satisfies LSI$(1,\lambda)$ for some λ). On the *a priori* space-time scales, we are forced to multiply this term by N. Thus, we lose any information about expected rates of convergence, in probability or expectation, for the force discrepancy term even at the initial time — though at the initial time the expected force is exactly the force appearing in the relativistic Vlasov-Poisson system.

Predominantly, most experts are of the opinion that the collective effects necessary for the emergence of Vlasov-type behavior only occur on large space-time scales compared to any scales set by an *a priori* set of units. It is generally thought that on such short scales discrete particle effects will overwhelm the longer range interactions needed to produce mean-field motion. However, as our discussion in the previous subsection made clear, the collective rVP_ϵ^- force term will contribute to the overall dynamics if one starts with *iid* initial data — whenever a limiting dynamics exists.

Yet, our results strongly point to the conclusion that the rVP_ϵ^- force term is not the sole force term contributing to the limiting dynamics. As discussed, we suspect that the discrete particle effects manifest in the form of a "collision" operator (e.g. of Lenard-Balescu type). Therefore, it seems that the actual dynamics is governed by a hybrid of the effects predicted in the traditional physics literature and those proposed in [KTZ08].

4.10.2 Future Directions

Below, we mention a few ideas for future research projects motivated by the results of the current chapter. First, discrete N-particle effects will likely lead to strongly erratic deviations from the expected value (which is the rVP^- force term). Therefore, one option would be to consider time-averages for some small window $(t-T, t+T)$ centered at time t. Of course, choosing an appropriate T will be tricky (and will certainly depend on N if not the specifics of an initial condition launching the dynamics). This procedure may also be of use in analyzing the two-specie relativistic Vlasov-Poisson system (4.67) launched by non-identical initial data f_0^+ and f_0^- for the different species. Here, rapid oscillations due to overall charge discrepancies will be the typical behavior rather than the exception. Also, we would eventually like to extend our work to hydrogenic plasmas (where the masses of the two species will be wildly different). In this case, even *iid* initial data jointly for both species will not lead to charge symmetry at any later times (as the lighter particles will move much more quickly on average). Here too, one would



(a) Slower Oscillation (b) Slow Oscillation

(c) Fast Oscillation (d) Faster Oscillation

Figure 4.1: Time averages of increasingly oscillatory functions

expect rapid oscillations of the lighter particle density about the density of its heavier companion species.

The situations we have in mind is not dissimilar to the zero-dispersion limit of certain PDEs (c.f. [LaLe79] and [EJLM03]). In these situation, when some parameter in the PDE is taken to zero the solutions develop very high frequency oscillations that certainly destroy strong convergence. Yet in some average or weak sense, the solutions still converge rather nicely. We give a schematic representation of this phenomenon in Figure 4.1.

Second, a possibility to characterize the "collision" operator is suggested by (4.122) and (4.123). Since in the critical case when $\kappa = 1$ (and so f_0 satisfies LSI$(1, \lambda)$ for some λ) the expected value of the magnitude of the force discrepancy converges to a constant and its variance is bounded, it may be that the discrepancy force term will converge in distribution with mean zero, and the departure of the particle dynamics from the Vlasov characteristics may be accurately modeled by some Markovian process. As discussed in Lancellotti's work on fluctuations of the non-relativistic Vlasov-Poisson system, this Markovian process could manifest itself as a Lenard-Balescu type collision operator [Lan10].

Once the issues above for the purely Coulombic dynamics have been sorted out, the ultimate goal is of course an investigation of the fully electromagnetic model with (approximate) spherical symmetry. This will certainly build upon the results presented here but will presumably entail more tricky estimates as the magnetic contributions to the overall force will no doubt be difficult to control. In fact, even should the issues for the Coulombic case be resolved without resorting to time-averaging, that technique may be indispensable in the fully electromagnetic model as deviations from sphericity could result in a significant magnetic field that only in some average sense will be zero.

4.11 Appendix: The Two-Body Problem and the Expected Build-Up of Correlations for Finite N-body Dynamics

In this appendix we illustrate the build-up of correlations using the two-body problem as a guiding example. Though two particles is far from the regime where we would expect any fluid model to be appropriate, it serves to demonstrate the build up of correlations rather well. First, let $T_{t,0}$ be the evolution of a system of N particles governed by a Hamiltonian H. Schematically, for a generic phase points $(\mathcal{P}, \mathcal{Q})$, we have

$$T_{t,0}(\mathcal{P}, \mathcal{Q}) = (\mathcal{P}(t), \mathcal{Q}(t)),$$

where

$$\begin{aligned}
\frac{d}{dt}\mathcal{P}(t) &= -\frac{\partial H}{\partial \mathcal{Q}}(\mathcal{P}(t), \mathcal{Q}(t)), \\
\frac{d}{dt}\mathcal{Q}(t) &= \frac{\partial H}{\partial \mathcal{P}}(\mathcal{P}(t), \mathcal{Q}(t)),
\end{aligned}$$

with initial condition $(\mathcal{P}(0), \mathcal{Q}(0)) = (\mathcal{P}, \mathcal{Q})$.

Define for any reasonably nice function g on the phase space

$$g_t(\mathcal{P},\mathcal{Q}) \equiv g \circ T_{t,0}(\mathcal{P},\mathcal{Q}) = (\mathcal{P}(t),\mathcal{Q}(t)).$$

As is well known (see any reference on Hamiltonian systems - e.g. Chapter 3 of [Th97]), the time evolution of the function g_t is given by the Poisson bracket:

$$\begin{aligned} \partial_t g_t &= \{g_t, H\}, \\ \{g_t, H\} &\equiv \frac{\partial g_t}{\partial \mathcal{Q}}\frac{\partial H}{\partial \mathcal{P}} - \frac{\partial g_t}{\partial \mathcal{P}}\frac{\partial H}{\partial \mathcal{Q}}. \end{aligned}$$

For our purposes, we are interested in the evolution of our probability measure $\mathbb{P}_0 \equiv \bigotimes_{i=2}^{2N} f_0\, d^3p\, d^3q$. That is, we would like to know the distribution of our ensemble of particles at time t given that they were distributed *iid* by f_0 at time 0. Let

$$F_0(p_1,\ldots,p_{2N},q_1,\ldots,q_{2N}) = \prod_{i=1}^{2N} f_0(p_i,q_i).$$

Since our dynamics are deterministic, the distribution of particles at time t is obtained simply by flowing a generic phase point of our system back to it's original condition and using F_0. That is

$$F_t(p_1,\ldots,p_{2N},q_1,\ldots,q_{2N}) = F_0 \circ T_{0,t}(p_1,\ldots,p_{2N},q_1,\ldots,q_{2N})$$

where $T_{0,t}$ is the flow associated with the negative of the Hamiltonian for our dynamics. This immediately leads to a PDE for the evolution of F_t:

$$\partial_t F_t + \{F_t, H\} = 0 \tag{4.125}$$

which is known as the *Liouville Equation.* To be very explicit, the relevant Liouville Equation for the dynamics given by (4.33) - (4.36) is

$$\partial_t F_t + \sum_{i=1}^{2N} \nabla_{q_i} F_t \cdot \frac{p_i}{\sqrt{1+|p_i|^2}} + e_i \nabla_{p_i} F_t \cdot \eta_\epsilon * E_t^\epsilon(q_i) = 0 \tag{4.126}$$

where e_i indicates the charge of the i-th particle and $\eta_\epsilon * E_t^\epsilon$ is the doubly regularized Coulomb force summed over all particles (we once again emphasize that the regularization causes the self-interaction term to be zero). Of course, this PDE is quite complicated! What we wish to emphasize is that even when F_0 is composed of a product over the individual particles (as we have assumed for our initial condition), at later times F_t need no longer be a product. In fact, it almost certainly will not be a product for any reasonable choice of f_0 (and hence our product function F_0). In other words, after the initial time, our particles will no longer be distributed independently. Since our proof replied heavily on independence or particles, there is no way to bootstrap the argument to later times.

Let us examine the situation for the two-body problem. Here, we can ignore the regularization of the force terms to make life a little easier. We can even begin with the *non-relativistic* two-body problem since the essential behavior is the same. So, we assume

$$F_0(p_1,p_2,q_1,q_2) = f_0(p_1,q_1)f_0(p_2,q_2)$$

with the evolution given by

$$\partial_t F_t + p_1\cdot\nabla_{q_1}F_t + p_2\cdot\nabla_{q_2}F_t - \frac{q_1-q_2}{|q_1-q_2|^3}\cdot\nabla_{p_1}F_t + \frac{q_1-q_2}{|q_1-q_2|^3}\cdot\nabla_{p_2}F_t \;=\; 0. \quad (4.127)$$

Though apparently daunting, this evolution does nothing but

$$\begin{aligned} F_t(p_1,p_2,q_1,q_2) =& f_0\left(P_1(p_1,p_2,q_1,q_2,t), Q_1(p_1,p_2,q_1,q_2,t)\right) \\ & \cdot f_0\left(P_2(p_1,p_2,q_1,q_2,t), Q_2(p_1,p_2,q_1,q_2,t)\right) \end{aligned}$$

where the functions P_i and Q_i satisfy

$$\begin{aligned} \partial_t Q_i &= -P_i, \\ \partial_t P_i &= (-1)^{i-1}\frac{Q_1-Q_2}{|Q_1-Q_2|^3}, \end{aligned}$$

with initial condition

$$(P_i,Q_i)(p_1,p_2,q_1,q_2,0) = (p_i,q_i).$$

Put simply, (P_1,P_2,Q_1,Q_2) is the phase point that evolves to (p_1,p_2,q_1,q_2) at time t (note that the coupled ODEs give the backwards flow). Clearly, the fact that these four ODEs are coupled implies that we lose the product structure in F_t for generic f_0.

Interestingly, the non-relativistic two-body problem will maintain the independence of the center-of-mass (COM) and relative coordinates if they are chosen independently at the initial time. Recall that these coordinates are given by

$$P \equiv p_1 + p_2 \qquad Q \equiv \frac{q_1+q_2}{2},$$
$$p \equiv \frac{p_1 - p_2}{2} \qquad q \equiv q_1 - q_2.$$

In these coordinates, the Liouville Equation becomes

$$\partial_t F_t + P\cdot\nabla_Q F_t + p\cdot\nabla_q F_t - \frac{q}{|q|^3}\cdot\nabla_p F_t \;=\; 0. \tag{4.128}$$

The absence of any term involving $\nabla_P F_t$ (which is a consequence of the Hamiltonian being independent of Q) indicates that the total linear momentum P is conserved. Note also that PDE has a strict separation between the terms involving the COM coordinates and those involving the relative coordinates. So, if

$$F_0(P,p,Q,q) = g(P,Q)h(p,q),$$

then for later times we have

$$F_t(P,p,Q,q) = g(P,Q-Pt)h\left(\tilde{p}(p,q,t),\tilde{q}(p,q,t)\right)$$

where $(\tilde{p},\tilde{q})$ is the initial condition for the effective one-body problem that will evolve into the point (p,q) at time t. This does not imply independence for the original coordinates (p_1,p_2,q_1,q_2)! Even though it may seem that choosing the center-of-mass and relative coordinates independently should be equivalent to choosing the original coordinates independently, this is not the case in general. All that we can say is that a function of the form $g(P,Q)h(p,q)$ is a linear superposition of functions of the form $\tilde{g}(p_1,q_1)\tilde{h}(p_2,q_2)$. Such a superposition will not in general correspond to independence of (p_1,q_1) and (p_2,q_2).

The two-body problem in our pseudo-relativistic setting is sufficiently different from the non-relativistic case to warrant some examination. However, the end result will be the same: independence of the two-particles will not in general propagate past the initial time. The relevant Hamiltonian for our considerations is

$$H(p_1,p_2,q_1,q_2) = \sqrt{1+|p_1|^2} + \sqrt{1+|p_2|^2} - |q_1-q_2|^{-1}.$$

As before, the coupling in the potential between q_1 and q_2 is directly responsible for the loss of independence at later times.

Let us examine the behavior in the COM and relative coordinates. The transformed Hamiltonian now reads:

$$H(P,p,q) = \sqrt{1 + |\frac{1}{2}P + p|^2} + \sqrt{1 + |\frac{1}{2}P - p|^2} - |q|^{-1}.$$

As in the non-relativistic case, the absence of Q in the Hamiltonian means that the total momentum P is conserved. However, the terms involving P and p now fail to separate! If we boos to the frame where $P = 0$ and $Q = 0$, then we do reduce to an effective one-body interaction as before. If we are in some frame where P is not zero, then the motion of the center-of-mass Q is no longer on a straight line determined by $Q(0)$ and P. The motion of Q will also depend on the relative momentum p! Hence, the pseudo-relativistic Hamiltonian above fails to preserve independence of the COM and relative coordinates as well!

For the sake of completeness, we also describe the orbits of the relative coordinate in the frame where $P = 0$. This has little to do with the major focus of this section (namely the failure of independence after the initial time), but it serves to reinforce the dramatic shift in behavior that occurs in relativistic dynamics. The Hamiltonian for the relative coordinates is given by

$$H(p,q) = 2\sqrt{1 + |p|^2} - |q|^{-1}.$$

Looking at the Hamiltonian above shows us that the angular momentum $L = q \times p$ is a conserved quantity just as in the non-relativistic case. Going through the same procedures as in the non-relativistic two-body problem, we find that when L is larger in magnitude than a certain threshold (1/2 in our system of units), then the orbits are similar to the classical ones (hyperbolae, parabolae, or ellipses depending on the energy). If the magnitude of the angular momentum is below this threshold and the energy is also below some other threshold (4 in our system of units), the coordinate q will spiral into the origin. Except perhaps at the boundary values, q will reach the origin in finite time! This is markedly different from the non-relativistic case where

q can never hit the origin as long as the initial angular momentum is non-zero (i.e. the particles are not moving directly towards or away from one another). Simply by replacing the non-relativistic velocity with its relativistic counterpart, it is possible to get system collapse in finite time for just two particles.

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