MTH 112 Spring 2021

Quiz 4 **Solutions:**

1.) Set up (but do not evaluate) an integral that would tell you how much work it would take to empty a spherical tank with radius 5 meters from the top. Let the density of the fluid be ρ kg/m^3. You can think of the tank as the curve $x = \sqrt{25 - y^2}$ on the interval $-5 \leq y \leq 5$.

Solution: We go through the usual analysis of the work done on a thin slab of thickness dy at height y.

- Volume of slab: $\pi r^2 h = \pi(\sqrt{25 - y^2})^2 dy = \pi(25 - y^2)dy$
- Mass of slab: (density)(volume) $= \rho\pi(25 - y^2)dy$
- Height slab moved: $5 - y$
- Work on slab: $mgh = \rho\pi g(25 - y^2)(5 - y)dy$

So, the total work to empty the tank from the top is given by

$$W = \int_{-5}^{5} \rho\pi g(25 - y^2)(5 - y)dy. \quad \blacksquare$$

2.) Consider the planar region contained between the curves $y = x^2 - 2$ and $y = 6 - x^2$. The area of this region is $\frac{64}{3}$ and the curves intersect at $x = \pm 2$.

a) Why is $\overline{x} = 0$ for the centroid of the region?

Solution: Since the region is symmetric about the y-axis and has constant density, the center of mass must be on that axis. $\blacksquare$

b) Find the y-coordinate of the centroid of this region (i.e. find $\overline{y}$).

Solution: We take the density to be 1, and so the total mass of the region is numerically the same as its area. Cutting the region into thin strips centered at x with thickness dx gives:

$$\begin{aligned} dm &= \left[(6 - x^2) - (x^2 - 2)\right] dx = (8 - 2x^2)dx, \\ \widetilde{x} &= x, \\ \widetilde{y} &= \frac{(6 - x^2) + (x^2 - 2)}{2} = 2. \end{aligned}$$

Notice that we are given

$$M = \int_{-2}^{2} dm = \int_{-2}^{2} (8 - 2x^2)dx = \frac{64}{3}.$$

For $\overline{y}$, we need to compute the x-moment:

$$\begin{aligned} M_x &= \int_{-2}^{2} \widetilde{y}\, dm \\ &= \int_{-2}^{2} 2(8-2x^2)dx \\ &= 2\int_{-2}^{2} (8-2x^2)dx \\ &= 2\left(\frac{64}{3}\right) \end{aligned}$$

where in the next to last step, we recognize that the integral we have to do is the same as the one for M! That means

$$\overline{y} = \frac{M_x}{M} = 2\left(\frac{64}{3}\right)\left(\frac{3}{64}\right) = 2. \quad \blacksquare$$