Summary of Hyperbolic Functions

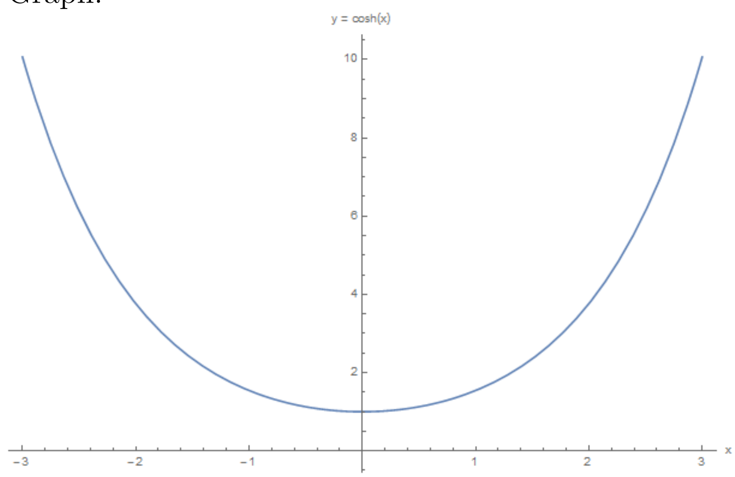
I. **Must-Know Information**

A. **Basic Definitions and Graphs**

1. **The Hyperbolic Cosine:** $\cosh(x)$

   a. Definition:
   $$\cosh(x) = \frac{e^x + e^{-x}}{2}$$

   b. Domain: $(-\infty, +\infty)$, Range: $[1, +\infty)$
   Graph:

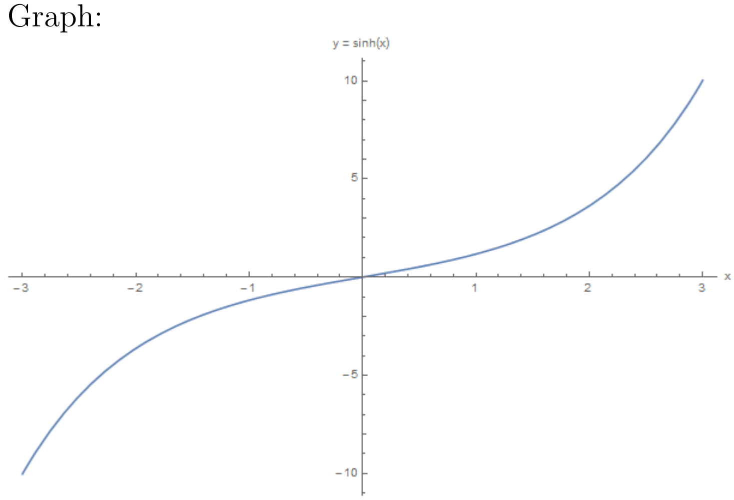


   c. Symmetry – Even: $\cosh(-x) = \cosh(x)$

2. **The Hyperbolic Sine:** $\sinh(x)$

   a. Definition:
   $$\sinh(x) = \frac{e^x - e^{-x}}{2}$$

   b. Domain: $(-\infty, +\infty)$, Range: $(-\infty, +\infty)$
   Graph:



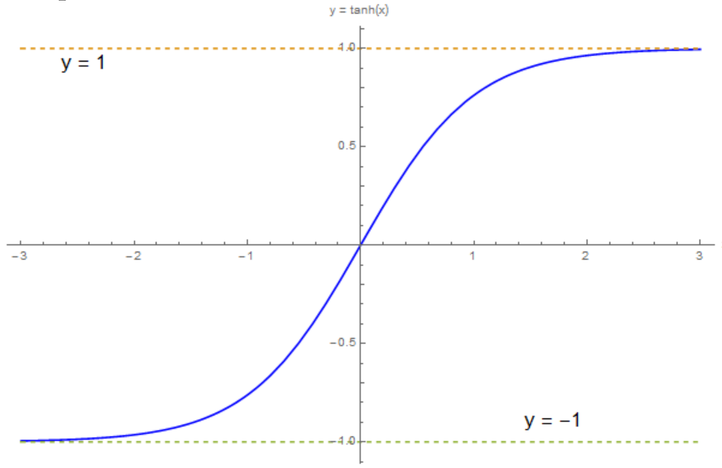
   c. Symmetry – Odd: $\sinh(-x) = -\sinh(x)$

3. **The Hyperbolic Tangent:** $\tanh(x)$

   a. Definition:

$$\tanh(x) = \frac{\sinh(x)}{\cosh(x)} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

   b. Domain: $(-\infty, \infty)$, Range: $(-1, 1)$
   (horizontal asymptotes at $y = -1$ and $y = 1$)
   Graph:



   c. Symmetry – Odd: $\tanh(-x) = -\tanh(x)$

4. **The Other Hyperbolic Functions**

   a. The Hyperbolic Secant: $\operatorname{sech}(x)$
   Definition: $\operatorname{sech}(x) = 1/\cosh(x)$
   Symmetry: Even

   b. The Hyperbolic Cosecant: $\operatorname{csch}(x)$
   Definition: $\operatorname{csch}(x) = 1/\sinh(x)$
   Symmetry: Odd

   a. The Hyperbolic Cotangent: $\coth(x)$
   Definition: $\coth(x) = 1/\tanh(x)$
   Symmetry: Odd

B. **Important Identities**

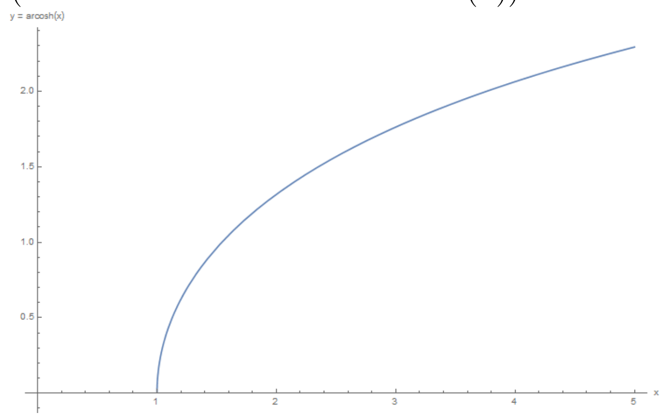
1. $\cosh^2(x) - \sinh^2(x) = 1$
2. $\sinh(2x) = 2\sinh(x)\cosh(x)$
3. $\cosh(2x) = \cosh^2(x) + \sinh^2(x)$
4. $\cosh^2(x) = \frac{1}{2}(\cosh(2x) + 1)$
5. $\sinh^2(x) = \frac{1}{2}(\cosh(2x) - 1)$

C. **Derivatives and Integrals**

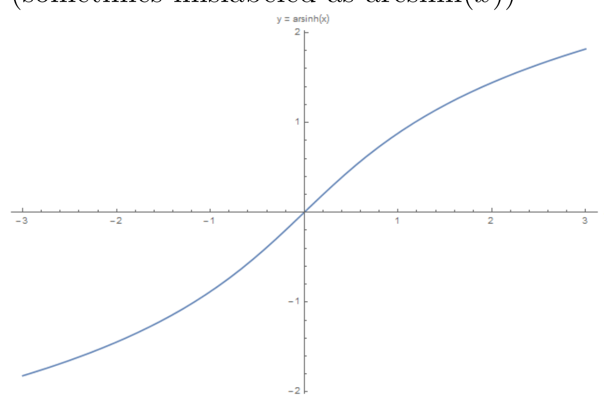
1. $\frac{d}{dx}[\cosh(x)] = \sinh(x), \qquad \int \cosh(x)\,dx = \sinh(x) + C$
2. $\frac{d}{dx}[\sinh(x)] = \cosh(x), \qquad \int \sinh(x)\,dx = \cosh(x) + C$
3. $\frac{d}{dx}[\tanh(x)] = \operatorname{sech}^2(x), \qquad \int \operatorname{sech}^2(x)\,dx = \tanh(x) + C$
   $\int \tanh(x)\,dx = \ln(\cosh(x)) + C$

D. **Inverse Hyperbolic Functions**

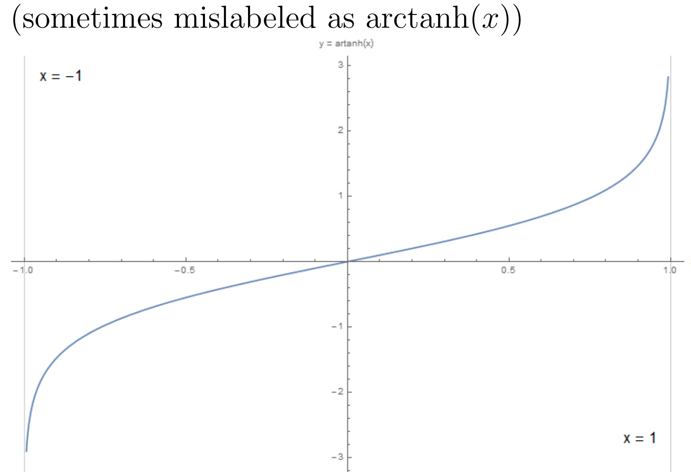
1. **The Inverse Hyperbolic Cosine:** $\cosh^{-1}(x)$ or $\operatorname{arcosh}(x)$
   Domain: $[1, +\infty)$, Range: $[0, +\infty)$, Symmetry: NONE
   (sometimes mislabeled as $\operatorname{arccosh}(x)$)



2. **The Inverse Hyperbolic Sine:** $\sinh^{-1}(x)$ or $\operatorname{arsinh}(x)$
   Domain: $(-\infty, +\infty)$, Range: $(-\infty, +\infty)$, Symmetry: Odd
   (sometimes mislabeled as $\operatorname{arcsinh}(x)$)



3. **The Inverse Hyperbolic Tangent:** $\tanh^{-1}(x)$ or $\operatorname{artanh}(x)$
   Domain: $(-1, 1)$, Range: $(-\infty, +\infty)$, Symmetry: Odd
   (sometimes mislabeled as $\operatorname{arctanh}(x)$)

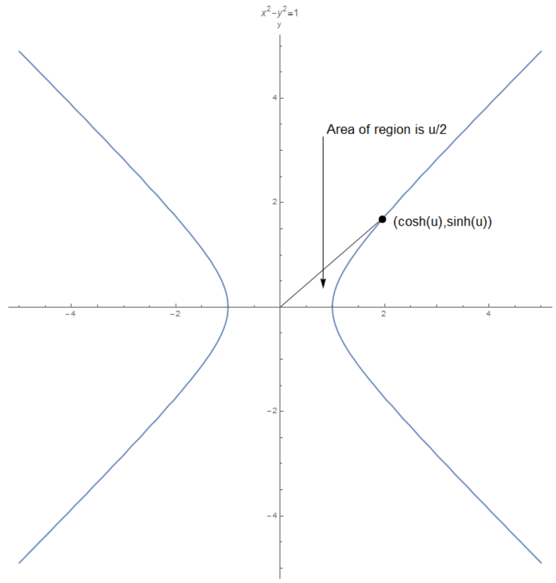


4. **Derivatives of the Inverse Hyperbolic Functions**

$f(x)$	$f'(x)$
$\operatorname{arcosh}(x)$	$\frac{1}{\sqrt{x^2-1}}$
$\operatorname{arsinh}(x)$	$\frac{1}{\sqrt{x^2+1}}$
$\operatorname{artanh}(x)$	$\frac{1}{1-x^2}$

II. **Additional Information to be Aware Of**

The name "hyperbolic" comes from the fact that these functions parametrize the right branch of the unit hyperbola $x^2 - y^2 = 1$.



The "hyperbolic angle" u is double the area enclosed by the hyperbola, the x-axis, and the ray from the origin to the point on the hyperbola parametrized by u. All of this is analogous to the definition of the usual trigonometric functions (which parametrize the unit circle $x^2+y^2=1$).

The Inverse Hyperbolic Functions all have formulae in terms of logarithms (not too surprising since they are all defined in terms of exponentials). We have the following equalities:

$$\begin{aligned}
\operatorname{arcosh}(x) &= \ln\left(x + \sqrt{x^2-1}\right) \text{ for } x \geq 1, \\
\operatorname{arsinh}(x) &= \ln\left(x + \sqrt{x^2+1}\right) \text{ for all real } x, \\
\operatorname{artanh}(x) &= \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) \text{ for } -1 < x < 1.
\end{aligned}$$

You do not need to memorize these, but you should be aware they exist since you will sometimes see computer algebra systems give integrals in terms of the inverse hyperbolic functions(where trig substitution will give the logarithmic forms). Also, you do not need these forms to deduce the derivatives of the inverse hyperbolic functions. For example,

$$\begin{aligned}
y = \operatorname{arsinh}(x) &\longleftrightarrow \sinh(y) = x \\
\cosh(y)\frac{dy}{dx} &= 1 \\
\frac{dy}{dx} &= \frac{1}{\cosh(y)} = \frac{1}{\cosh(\operatorname{arsinh}(x))} = \frac{1}{\sqrt{x^2+1}}
\end{aligned}$$

For the last equality, we use $\cosh^2(x) - \sinh^2(x) = 1$ to find that $\cosh(x) = \sqrt{\sinh^2(x)+1}$ (recall that $\cosh(x)$ is always positive). Hence

$$\cosh(\operatorname{arsinh}(x)) = \sqrt{\sinh(\operatorname{arsinh}(x))^2+1} = \sqrt{x^2+1}.$$