Review of Vector Calculus

1 Differential Operators

1.1 First–Order Operators

For a real–valued function of two variables, $f : \mathbb{R}^2 \to \mathbb{R}$, the **gradient** of f is given by

$$\nabla f = \frac{\partial f}{\partial x}\hat{\boldsymbol{\imath}} + \frac{\partial f}{\partial y}\hat{\boldsymbol{\jmath}}.$$

For a real–valued function of three variables, $f : \mathbb{R}^3 \to \mathbb{R}$, the **gradient** of f is given by

$$\nabla f = \frac{\partial f}{\partial x}\hat{\boldsymbol{\imath}} + \frac{\partial f}{\partial y}\hat{\boldsymbol{\jmath}} + \frac{\partial f}{\partial z}\hat{\boldsymbol{k}}.$$

In both cases, note that the gradient operator, ∇, takes as argument a *scalar–valued* function and returns a *vector field*.

For vector fields, $\vec{\boldsymbol{F}} : \mathbb{R}^3 \to \mathbb{R}^3$, there are two major first–order differential operators. Suppose that

$$\vec{\boldsymbol{F}}(x, y, z) = P(x, y, z)\hat{\boldsymbol{\imath}} + Q(x, y, z)\hat{\boldsymbol{\jmath}} + R(x, y, z)\hat{\boldsymbol{k}}.$$

The **divergence** of $\vec{\boldsymbol{F}}$ is given by

$$\nabla \cdot \vec{\boldsymbol{F}} = \left(\frac{\partial}{\partial x}\hat{\boldsymbol{\imath}} + \frac{\partial}{\partial y}\hat{\boldsymbol{\jmath}} + \frac{\partial}{\partial z}\hat{\boldsymbol{k}}\right) \cdot \left(P\hat{\boldsymbol{\imath}} + Q\hat{\boldsymbol{\jmath}} + R\hat{\boldsymbol{k}}\right) = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}.$$

Note that the divergence operator, $\nabla\cdot$, takes as argument a *vector field* and returns a *scalar–valued* function.

The **curl** of $\vec{\boldsymbol{F}}$ is given by

$$\begin{aligned}\nabla \times \vec{\boldsymbol{F}} &= \begin{vmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\boldsymbol{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} \\ &= (R_y - Q_z)\,\hat{\boldsymbol{\imath}} + (P_z - R_x)\,\hat{\boldsymbol{\jmath}} + (Q_x - P_y)\,\hat{\boldsymbol{k}}.\end{aligned}$$

Note that the curl operator, $\nabla\times$, takes as argument a *vector field* and returns a *vector field*. Both the divergence and curl have obvious restrictions to vectors fields in $\mathbb{R}^2$.

1.2 Second–Order Operators

The **Laplacian** of a real–valued function $f : \mathbb{R}^3 \to \mathbb{R}$ is given by

$$\triangle f = \nabla^2 f = \nabla \cdot \nabla f = f_{xx} + f_{yy} + f_{zz}$$

(with the obvious restriction for functions of two variables).

There are two identities worth mentioning here (which are true for "nice enough" functions [1])

$$\nabla \times \nabla f = \vec{0} \quad \text{(curl of gradient is zero)},$$
$$\nabla \cdot (\nabla \times \vec{\boldsymbol{F}}) = 0 \quad \text{(divergence of curl is zero)}.$$

Both of these boil down to the property that mixed partial derivatives are always equal for nice functions.

1.3 Some Useful Identities

There are four useful vector identities that hold true for any three–vectors $\vec{\boldsymbol{a}}, \vec{\boldsymbol{b}}, \vec{\boldsymbol{c}}$, and $\vec{\boldsymbol{d}}$.

1. $\vec{\boldsymbol{a}} \times \vec{\boldsymbol{b}} = -\left(\vec{\boldsymbol{b}} \times \vec{\boldsymbol{a}}\right)$
2. $\vec{\boldsymbol{a}} \cdot \left(\vec{\boldsymbol{b}} \times \vec{\boldsymbol{c}}\right) = \vec{\boldsymbol{b}} \cdot \left(\vec{\boldsymbol{c}} \times \vec{\boldsymbol{a}}\right) = \vec{\boldsymbol{c}} \cdot \left(\vec{\boldsymbol{a}} \times \vec{\boldsymbol{b}}\right)$
   (NOTE: The three vectors are cyclically permuted in the three different forms!)
3. $\vec{\boldsymbol{a}} \times \left(\vec{\boldsymbol{b}} \times \vec{\boldsymbol{c}}\right) = \left(\vec{\boldsymbol{a}} \cdot \vec{\boldsymbol{c}}\right)\vec{\boldsymbol{b}} - \left(\vec{\boldsymbol{a}} \cdot \vec{\boldsymbol{b}}\right)\vec{\boldsymbol{c}}$
4. $\left(\vec{\boldsymbol{a}} \times \vec{\boldsymbol{b}}\right) \cdot \left(\vec{\boldsymbol{c}} \times \vec{\boldsymbol{d}}\right) = \left(\vec{\boldsymbol{a}} \cdot \vec{\boldsymbol{c}}\right)\left(\vec{\boldsymbol{b}} \cdot \vec{\boldsymbol{d}}\right) - \left(\vec{\boldsymbol{a}} \cdot \vec{\boldsymbol{d}}\right)\left(\vec{\boldsymbol{b}} \cdot \vec{\boldsymbol{c}}\right)$

These vector identities give rise to some useful identities for our differential operators. Let $\vec{\boldsymbol{F}}$ and $\vec{\boldsymbol{G}}$ be vector fields, and let ψ be a scalar–valued function.

A. $\nabla \cdot \left(\psi \vec{\boldsymbol{F}}\right) = \nabla \psi \cdot \vec{\boldsymbol{F}} + \psi \nabla \cdot \vec{\boldsymbol{F}}$ (product rule for divergence)

B. $\nabla \times \left(\psi \vec{\boldsymbol{F}}\right) = \nabla \psi \times \vec{\boldsymbol{F}} + \psi \nabla \times \vec{\boldsymbol{F}}$ (product rule for curl)

C. $\nabla \cdot \left(\vec{\boldsymbol{F}} \times \vec{\boldsymbol{G}}\right) = \left(\nabla \times \vec{\boldsymbol{F}}\right) \cdot \vec{\boldsymbol{G}} - \vec{\boldsymbol{F}} \cdot \left(\nabla \times \vec{\boldsymbol{G}}\right)$

D. $\nabla \times \left(\vec{\boldsymbol{F}} \times \vec{\boldsymbol{G}}\right) = \left(\nabla \cdot \vec{\boldsymbol{G}}\right)\vec{\boldsymbol{F}} - \left(\nabla \cdot \vec{\boldsymbol{F}}\right)\vec{\boldsymbol{G}} + \left(\vec{\boldsymbol{G}} \cdot \nabla\right)\vec{\boldsymbol{F}} - \left(\vec{\boldsymbol{F}} \cdot \nabla\right)\vec{\boldsymbol{G}}$

E. $\nabla\left(\vec{\boldsymbol{F}} \cdot \vec{\boldsymbol{G}}\right) = \left(\vec{\boldsymbol{F}} \cdot \nabla\right)\vec{\boldsymbol{G}} + \left(\vec{\boldsymbol{G}} \cdot \nabla\right)\vec{\boldsymbol{F}} + \vec{\boldsymbol{F}} \times \left(\nabla \times \vec{\boldsymbol{G}}\right) + \vec{\boldsymbol{G}} \times \left(\nabla \times \vec{\boldsymbol{F}}\right)$

[1] Here, "nice enough" basically means that all possible second–partial derivatives exist and are *continuous*.

The last two identities require some explanation of the notation. If

$$\vec{\boldsymbol{F}} = P\hat{\boldsymbol{\imath}} + Q\hat{\boldsymbol{\jmath}} + R\hat{\boldsymbol{k}} \text{ and } \vec{\boldsymbol{G}} = U\hat{\boldsymbol{\imath}} + V\hat{\boldsymbol{\jmath}} + W\hat{\boldsymbol{k}},$$

then the notation $\vec{\boldsymbol{G}} \cdot \nabla$ stands for the differential operator

$$\vec{\boldsymbol{G}} \cdot \nabla = U\frac{\partial}{\partial x} + V\frac{\partial}{\partial y} + W\frac{\partial}{\partial z}.$$

So, the term $\left(\vec{\boldsymbol{G}} \cdot \nabla\right) \vec{\boldsymbol{F}}$ stands for

$$\begin{aligned}\left(\vec{\boldsymbol{G}} \cdot \nabla\right) \vec{\boldsymbol{F}} &= \left(U\frac{\partial}{\partial x} + V\frac{\partial}{\partial y} + W\frac{\partial}{\partial z}\right)\left(P\hat{\boldsymbol{\imath}} + Q\hat{\boldsymbol{\jmath}} + R\hat{\boldsymbol{k}}\right) \\ &= \left(U\frac{\partial P}{\partial x} + V\frac{\partial P}{\partial y} + W\frac{\partial P}{\partial z}\right)\hat{\boldsymbol{\imath}} + \left(U\frac{\partial Q}{\partial x} + V\frac{\partial Q}{\partial y} + W\frac{\partial Q}{\partial z}\right)\hat{\boldsymbol{\jmath}} \\ &\qquad + \left(U\frac{\partial R}{\partial x} + V\frac{\partial R}{\partial y} + W\frac{\partial R}{\partial z}\right)\hat{\boldsymbol{k}}\end{aligned}$$

2 Main Theorems of Vector Calculus

2.1 Line and Surface Integrals in $\mathbb{R}^3$

If $\mathcal{C}$ is a (sufficiently nice) space-curve, and f is a scalar–valued function defined over that curve, then the **line integral** of f over $\mathcal{C}$ is given by

$$\int_{\mathcal{C}} f\ dr = \int_a^b f(\vec{\boldsymbol{r}}(t))\left|\vec{\boldsymbol{r}}\,'(t)\right|\ dt$$

where $\vec{\boldsymbol{r}}(t) = x(t)\hat{\boldsymbol{\imath}} + y(t)\hat{\boldsymbol{\jmath}} + z(t)\hat{\boldsymbol{k}}$ is any parametrization of $\mathcal{C}$ (starting at $t = a$ and ending at $t = b$). The notation $f(\vec{\boldsymbol{r}}(t))$ is shorthand for

$$f(\vec{\boldsymbol{r}}(t)) = f(x(t), y(t), z(t)).$$

If $\vec{\boldsymbol{F}}$ is a vector field defined over the curve, then the integral of this vector field over $\mathcal{C}$ is defined to be

$$\int_{\mathcal{C}} \vec{\boldsymbol{F}} \cdot d\vec{\boldsymbol{r}} = \int_a^b \vec{\boldsymbol{F}}(\vec{\boldsymbol{r}}(t)) \cdot \vec{\boldsymbol{r}}\,'(t)\ dt.$$

If $\mathcal{C}$ is a simple, closed curve, you often see the notation

$$\oint_{\mathcal{C}} f\ dr \text{ and } \oint_{\mathcal{C}} \vec{\boldsymbol{F}} \cdot d\vec{\boldsymbol{r}}$$

for these integrals. The definitions are entirely the same, and the circle on the integral sign only emphasizes that the curve is closed. The integral of a vector field over a simple closed curve is often called the **circulation** of the field around the curve.

If $\mathcal{S}$ is a surface in $\mathbb{R}^3$, then a parametrization of the surface is a function

$$\vec{\boldsymbol{r}}(u,v) = x(u,v)\hat{\boldsymbol{\imath}} + y(u,v)\hat{\boldsymbol{\jmath}} + z(u,v)\hat{\boldsymbol{k}}$$

where u and v range over some two–dimensional region $\mathcal{R}$. If f is a scalar–valued function defined over the surface, then we define the **surface integral** of f over $\mathcal{S}$ to be

$$\iint_{\mathcal{S}} f \, d\sigma = \iint_{\mathcal{R}} f(\vec{\boldsymbol{r}}(u,v)) \; |\vec{\boldsymbol{r}}_u \times \vec{\boldsymbol{r}}_v| \; dudv.$$

If $\mathcal{S}$ is an *orientable* surface, then it is possible to choose a consistent normal vector at each point on the surface. We will assume that this normal vector is given by $\vec{\boldsymbol{r}}_u \times \vec{\boldsymbol{r}}_v$ in our parametrization. Then the **flux** of a vector field $\vec{\boldsymbol{F}}$ through the surface $\mathcal{S}$ is defined to be

$$\iint_{\mathcal{S}} \vec{\boldsymbol{F}} \cdot d\vec{\boldsymbol{\sigma}} = \iint_{\mathcal{R}} \vec{\boldsymbol{F}}(\vec{\boldsymbol{r}}(u,v)) \cdot (\vec{\boldsymbol{r}}_u \times \vec{\boldsymbol{r}}_v) \; dudv.$$

2.2 The Fundamental Theorem for Line Integrals

Let $\mathcal{C}$ be any piecewise smooth curve with endpoints $P_0 = (x_0, y_0, z_0)$ and $P_1 = (x_1, y_1, z_1)$ and oriented so that the curve starts at P_0 and ends at P_1. Then for any function f with continuous first–partials,

$$\int_{\mathcal{C}} \nabla f \cdot d\vec{\boldsymbol{r}} = f(P_1) - f(P_0).$$

Notice that the integral does not depend on the precise details of the path $\mathcal{C}$, only the endpoints.

If a vector field $\vec{\boldsymbol{F}}$ is defined on a *simply connected* open domain and the components of $\vec{\boldsymbol{F}}$ have continuous partial derivatives, then $\vec{\boldsymbol{F}} = \nabla f$ if and only if

$$\nabla \times \vec{\boldsymbol{F}} = \vec{0}.$$

In such case, we say that f is a **potential function** for the vector field $\vec{\boldsymbol{F}}$.

2.3 Stokes' Theorem

Let $\mathcal{S}$ be a piecewise smooth, oriented surface that is bounded by a simple, closed, piecewise smooth curve $\mathcal{C}$ given the induced orientation from $\mathcal{S}$. Suppose a vector field $\vec{\boldsymbol{F}}$ has continuous partial derivatives on an open region containing $\mathcal{S}$. Then

$$\oint_{\mathcal{C}} \vec{\boldsymbol{F}} \cdot d\vec{\boldsymbol{r}} = \iint_{\mathcal{S}} \left(\nabla \times \vec{\boldsymbol{F}}\right) \cdot d\vec{\boldsymbol{\sigma}}.$$

2.4 The Divergence Theorem

Let $\mathcal{E}$ be a simple solid region in space that is bounded by a closed surface $\mathcal{S}$ given the outward orientation. Suppose a vector field $\vec{\boldsymbol{F}}$ has continuous partial derivatives on an open region containing $\mathcal{E}$. Then

$$\oiint_{\mathcal{S}} \vec{\boldsymbol{F}} \cdot d\vec{\boldsymbol{\sigma}} = \iiint_{\mathcal{E}} \left(\nabla \cdot \vec{\boldsymbol{F}}\right)\, dV.$$

As with line integrals, the circle on the surface integral indicates a *closed surface*, i.e. a surface with no boundary (like a sphere).

2.5 Green's Identities

Suppose that $\vec{\boldsymbol{G}} = u\vec{\boldsymbol{F}}$ where u is a differentiable scalar–valued function and $\vec{\boldsymbol{F}}$ is a differentiable vector field. If $\mathcal{E}$ is a simple solid region in space that is bounded by a closed surface $\mathcal{S}$, then by the Divergence Theorem we know that

$$\iiint_{\mathcal{E}} \nabla \cdot \vec{\boldsymbol{G}}\, dV = \oiint_{\mathcal{S}} \vec{\boldsymbol{G}} \cdot d\vec{\boldsymbol{\sigma}}.$$

From the product rule for the divergence operator, we have

$$\nabla \cdot \vec{\boldsymbol{G}} = \nabla u \cdot \vec{\boldsymbol{F}} + u\nabla \cdot \vec{\boldsymbol{F}}.$$

If we substitute this identity and let $\vec{\boldsymbol{F}} = \nabla v$, we have

$$\iiint_{\mathcal{E}} \nabla u \cdot \nabla v + u \triangle v\, dV = \oiint_{\mathcal{S}} u\nabla v \cdot d\vec{\boldsymbol{\sigma}}.$$

This identity (which is true for any twice–differentiable functions u and v) is known as *Green's First Identity.*

If we take Green's First Identity and interchange the roles of u and v, we obtain an equivalent version of the identity:

$$\iiint_{\mathcal{E}} \nabla v \cdot \nabla u + v \triangle u\, dV = \oiint_{\mathcal{S}} v\nabla u \cdot d\vec{\boldsymbol{\sigma}}.$$

If we subtract this new version from the original, we arrive at *Green's Second Identity*

$$\iiint_{\mathcal{E}} (u\triangle v - v\triangle u)\, dV = \oiint_{\mathcal{S}} (u\nabla v - v\nabla u) \cdot d\vec{\boldsymbol{\sigma}}.$$