MTH 112 : Spring 2021
Formula Packet for Exam 4

- Important Area Formulas
  - The area of a disk of radius r is given by $A = \pi r^2$.
  - The area of an annulus (or washer) with outer radius R and inner radius r is given by
  $$A = \pi \left[R^2 - r^2\right].$$
  - The surface area of a cylindrical shell with radius r and height h is given by
  $$A = 2\pi r h.$$
- Arc Length and Surface Area
  - The **arc length**, L, of a curve $y = f(x)$ on an interval $[a, b]$ is given by
  $$L = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx,$$
  provided the derivative is continuous on (a, b).
  - The **surface area**, S, generated by revolving the graph of $y = f(x)$ (for $f(x) \geq 0$) on an interval $[a, b]$ about the x-axis is given by
  $$S = 2\pi \int_a^b f(x)\sqrt{1 + [f'(x)]^2}\, dx,$$
  provided the derivative is continuous on (a, b).
- Work
  - The total work done by a variable force $F = F(x)$ moving an object from $x = a$ to $x = b$ is
  $$W = \int_a^b F(x)dx.$$
  - The work done against gravity (near the surface of a planet) in moving an object with mass m vertically upward a distance h is $W = mgh$ where g is the acceleration due to gravity near the surface.
- Center of Mass
  - The center of mass for a planar figure is the point $(\overline{x}, \overline{y})$ where
  $$\overline{x} = \frac{M_y}{M},$$
  $$\overline{y} = \frac{M_x}{M}.$$

- In the formulas above,

$$M = \int_a^b \, dm,$$
$$M_y = \int_a^b \widetilde{x} \, dm,$$
$$M_x = \int_a^b \widetilde{y} \, dm,$$

  where dm is the mass of an (infinitesimal) strip and $(\widetilde{x}, \widetilde{y})$ is the center of the strip.

- Half Angle Formulas
  - $\sin^2(x) = \frac{1}{2}(1 - \cos(2x))$
  - $\cos^2(x) = \frac{1}{2}(1 + \cos(2x))$
  - $\sinh^2(x) = \frac{1}{2}(\cosh(2x) - 1)$
  - $\cosh^2(x) = \frac{1}{2}(\cosh(2x) + 1)$

- **The Trapezoid Rule**:

$$T_n = \frac{\Delta x}{2}\left(y_0 + 2y_1 + 2y_2 + \cdots + 2y_{n-1} + y_n\right),$$

$$E_T \leq \frac{M_2(b-a)^3}{12n^2}$$

- **Simpson's Rule** (n must be even):

$$S_n = \frac{\Delta x}{3}\left(y_0 + 4y_1 + 2y_2 + 4y_3 + \cdots + 2y_{n-2} + 4y_{n-1} + y_n\right),$$

$$E_S \leq \frac{M_4(b-a)^5}{180n^4}$$

- For an infinite series $\sum a_n$:
  - **The Divergence Test (or N-th Term Test)**: If the sequence of terms does not converge to 0 (i.e. $\lim_{n\to\infty} a_n \neq 0$), then the infinite series diverges.
  - **The Integral Test**: If the terms of the series are given by $a_n = f(n)$ where f is a *positive, continuous, and decreasing* function over the range of summation, then either

$$\sum_{n=N}^{\infty} a_n \qquad \text{and} \qquad \int_N^{\infty} f(x)\,dx$$

    both converge or both diverge. Moreover, if the series converges, we have the estimate

$$S_n + \int_{n+1}^{\infty} f(x)\,dx \leq \sum_{n=N}^{\infty} a_n \leq S_n + \int_n^{\infty} f(x)\,dx,$$

    where S_n is the n-th partial sum ($n > N$).
  - **The Direct Comparison Test**: Suppose we have two non-negative sequences a_n and b_n satisfying $0 \leq a_n \leq b_n$ for all $n \geq N$.
    (a) If $\sum b_n$ converges, then so does $\sum a_n$.
    (b) If $\sum a_n$ diverges, then so does $\sum b_n$.
  - **The Limit Comparison Test**: Suppose we have two positive sequences $a_n > 0$ and $b_n > 0$.
    (a) If $\lim_{n\to\infty} a_n/b_n = c$ where $0 < c < \infty$, then $\sum a_n$ and $\sum b_n$ either both converge or both diverge.
    (b) If $\lim_{n\to\infty} a_n/b_n = 0$ AND $\sum b_n$ converges, then $\sum a_n$ also converges.
    (c) If $\lim_{n\to\infty} a_n/b_n = \infty$ AND $\sum b_n$ diverges, then $\sum a_n$ also diverges.
  - **The Ratio Test**: Suppose we have a sequence a_n of positive terms (i.e. $a_n > 0$) and

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = L.$$

    (a) If $0 \leq L < 1$, then $\sum a_n$ converges.
    (b) If $L > 1$, then $\sum a_n$ diverges.
    (c) If $L = 1$, the test is inconclusive.
  - **The Root Test**: Suppose we have a sequence a_n of positive terms (i.e. $a_n > 0$) and

$$\lim_{n\to\infty} \sqrt[n]{a_n} = L.$$

    (a) If $0 \leq L < 1$, then $\sum a_n$ converges.
    (b) If $L > 1$, then $\sum a_n$ diverges.
    (c) If $L = 1$, the test is inconclusive.

- **Absolute Convergence**: If $\sum |a_n|$ converges, then so does $\sum a_n$.
- **The Alternating Series Test:** Suppose we have a positive sequence $a_n > 0$ where the terms are non-increasing ($a_n \geq a_{n+1}$) and limit to zero ($a_n \to 0$). Then the series

$$\sum_{n=1}^{\infty} (-1)^{n-1} a_n = a_1 - a_2 + a_3 - a_4 + \cdots$$

converges. Moreover, if S_n is the n-th partial sum of the series above, then for any $N > 0$ we have

$$S_{2N} \leq \sum_{n=1}^{\infty} (-1)^{n-1} a_n \leq S_{2N+1}.$$

- **Taylor's Remainder Theorem**: Let P_n be the Taylor Polynomial of order n centered at c for some given function f. Suppose that M_{n+1} is a positive constant so that $|f^{(n+1)}(t)| \leq M_{n+1}$ for all t between c and x (inclusive). Then the difference (or remainder) between $f(x)$ and $P_n(x)$ satisfies

$$|R_n(x)| = |f(x) - P_n(x)| \leq M_{n+1} \frac{|x - c|^{n+1}}{(n+1)!}.$$