Computer Project 8
Linear Resistance vs. Sliding Friction

DUE: March 24, 2023

Introduction: When we studied Spring–Mass Systems earlier in the course, we used a linear model for the resistive force,

$$F_{\text{res}} = -bv,$$

where v is the velocity of the mass. Pairing that with a linear model for the spring force gave us the second–order ODE

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = 0.$$

The issue with this is that *sliding* friction is not actually a linear function of velocity! The simplest model of sliding friction is that it is proportional to the normal force (which is the force of contact between the mass and surface it is sliding on) and opposite to the velocity. More precisely, we should have

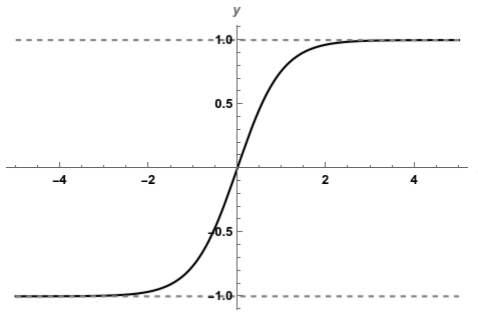
$$F_{\text{res}} = \left\{ \begin{array}{rl} -b, & v > 0 \\ b, & v < 0 \end{array} \right. .$$

We will not be too concerned about what happens when $v = 0$, though usually objects have a different coefficient of resistance when they are at rest.

The issue with using this model of resistance in our numerical scheme is that it is discontinuous. So, we are not necessarily guaranteed that RK–4 will be well-behaved for such a function. To make the model slightly nicer for computational purposes, we will replace the step function above with a continuous version. To that end, we will model friction using the hyperbolic tangent,

$$\tanh(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}.$$

This function has horizontal asymptotes of $y = \pm 1$ and looks similar to a simple step function. This function is pre-programmed in the `math` package in Python and can be called as `math.tanh(x)`.

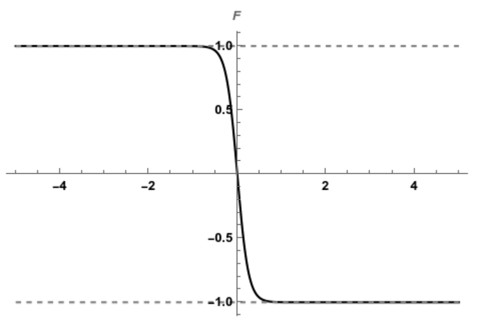


Plot of $y = \tanh(x)$

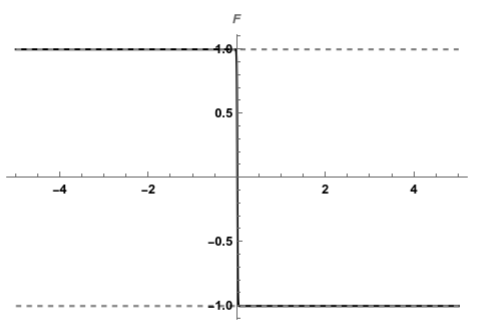
So, we will model sliding friction by

$$F_{\text{res}} = -b \tanh\left(\frac{v}{v_0}\right),$$

where v_0 effectively sets the scale where the transition from -1 to 1 occurs.



Plot of $F = -\tanh\left(\frac{v}{0.25}\right)$



Plot of $F = -\tanh\left(\frac{v}{0.01}\right)$

Instructions: Your goal in this project is to explore the differences in behavior between a spring–mass system with linear damping force versus non-linear sliding friction. We will take $m = 1$, $k = 1$, and $v_0 = 0.1$ for simplicity.

Linear Damping

$$\frac{d^2x}{dt^2} + b\frac{dx}{dt} + x = 0$$
$$x(0) = x_0$$
$$x'(0) = 0$$

Non-Linear Damping

$$\frac{d^2x}{dt^2} + b\tanh\left(10 \cdot \frac{dx}{dt}\right) + x = 0$$
$$x(0) = x_0$$
$$x'(0) = 0$$

1) Convert both the linear and non-linear systems above into systems of 2 first-order differential equations.

2) Program the right–hand sides of both systems into Python so that we can call our implementation of RK–4 for each.

3) Explore the differences in the two models by producing plots of the the solution $x(t)$ for the following combinations of b and x_0.

   a) $b = 0.1, x_0 = 1$
   b) $b = 0.1, x_0 = 5$
   c) $b = 0.4, x_0 = 1$
   d) $b = 0.4, x_0 = 5$

   For a) and b), you will probably need to go out to $t = 100$ (I used 10000 steps which seemed ok). For c) and d), you only need to go out to $t = 40$ to get a good sense of the behavior.

4) Write up a report based on your observations (1 to 2 pages should be enough). Be sure to comment on the effect of the non-linear damping on frequency of oscillation and how long it takes for the system to approach equilibrium. Try to explain what you are seeing! Feel free to explore other combinations of the parameters in the model!