Computer Project 5
Investigating Torricelli's Law

DUE: December, 19, 2022

Instructions: Suppose we have a tank of constant cross section (e.g. a cylindrical tank) where we are pumping fluid into the tank at a constant rate and simultaneously allowing the tank to drain by gravity.

According to Torricelli's Law, the height of fluid in the tank, $h(t)$, will satisfy

$$\frac{dh}{dt} = A - k\sqrt{h}$$
$$h(0) = h_0$$

where A is the constant flow rate into the tank divided by the cross sectional area, k is a physical parameter related to the strength of gravity and the size of the drain hole, and h_0 is the initial height of fluid in the tank. Note that this is a separable first order differential equation, but the solution is not expressible in terms of elementary functions (unless $A = 0$). We do know that the solution should approach an equilibrium value.

$$\lim_{t \to \infty} h(t) = \left(\frac{A}{k}\right)^2$$

The goal of this project is to explore how changing A, k, and h_0 changes the rate at which the solution approaches its equilibrium value.

First, we know that our solutions will never actually reach their equilibrium values (due to uniqueness of solutions). So given parameters A, k, and h_0, we will be interested in the time it takes the solution to get within 1% of the equilibrium value. If h_0 is less than $0.99(A/k)^2$, we want to find the time $T = T(A, k, h_0)$ so that $h(T) = 0.99(A/k)^2$. Likewise, if h_0 is greater than $1.01(A/k)^2$, we want to find the time $T = T(A, k, h_0)$ so that $h(T) = 1.01(A/k)^2$. If $0.99(A/k)^2 \le h_0 \le 1.01(A/k)^2$, we will take $T = 0$ since the solution is already sufficiently close to the equilibrium value. Note, the choice of a 1% window is completely arbitrary!

Instructions:

1) For each of the values of A, k, and h_0 below, produce a plot of the solution to the initial value problem on the previous page. Also, find the time when the solution is within 1% of the equilibrium solution. For each, running RK-4 over $0 \leq t \leq 10$ in 10,000 steps should be adequate.

    a) $A = 1$, $k = 1$, $h_0 = 0$

    b) $A = 1$, $k = 1$, $h_0 = 2$

    c) $A = 6$, $k = 3$, $h_0 = 0$

2) Now suppose that $A = 1$ and $k = 1$. We want to explore how long it will take the tank to reach equilibrium (within our 1% tolerance) for different initial heights. So for all initial heights h_0 in the range $1.02 \leq h_0 \leq 3$ in steps of 0.02, determine the first time when the solution falls below 1.01. Produce a plot of these times versus the initial height h_0. You will probably need to let t range over $0 \leq t \leq 12$ to get all of the data points. Keeping the number of steps at 10,000 will probably be fine (though more steps is better if it doesn't take forever to finish).

3) Now suppose that $A = 1$ and $h_0 = 0$. We want to explore how the time it takes solutions to approach equilibrium changes with k (effectively, increasing k corresponds to increasing the size of the drain hole). So, for all k in range $1 \leq k \leq 2$ in steps of 0.1, determine the first time when the solution rises to within 99% of the equilibrium value (which changes with k). Produce a plot of these times versus k. For each run, $0 \leq t \leq 10$ in 10,000 steps should be adequate.