Intro. to ODEs

Quiz 14 **Solutions**

1) Find and classify all equilibria for the 2×2 non-linear system below.

$$\begin{aligned}\frac{dx}{dt} &= xy - x - 4 \\ \frac{dy}{dt} &= x(y - x + 2)\end{aligned}$$

The easiest nullclines to work with are the ones for y.

$$\begin{aligned}x(y - x + 2) &= 0 \\ x = 0 \text{ or } y &= x - 2\end{aligned}$$

If $x = 0$, then the x–nullcline gives us $-4 = 0$ which is nonsense. So, there are no equilibria associated to this case. If $y = x - 2$, then the x-nullcline gives

$$\begin{aligned}xy - x - 4 &= 0 \\ x(x-2) - x - 4 &= 0 \\ x^2 - 3x - 4 &= 0 \\ (x-4)(x+1) &= 0 \\ x &= -1, 4.\end{aligned}$$

This gives us two equilibria: $(-1, -3)$ and $(4, 2)$. The linearization matrix is

$$L(x, y) = \begin{bmatrix} y - 1 & x \\ y - 2x + 2 & x \end{bmatrix}$$

which only has significance in a small neighborhood around each equilibrium.

- For $(-1, -3)$, $\tau = \operatorname{tr}(L(-1, -3)) = -5$ while $\Delta = \det(L(-1, -3)) = 5$. Since $\tau^2/4 > \Delta$, $\Delta > 0$, and $\tau < 0$, we know that $(-1, -3)$ is an *asymptotically stable node.*
- For $(4, 2)$, $\tau = \operatorname{tr}(L(4, 2)) = 5$ while $\Delta = \det(L(4, 2)) = 20$. Since $\tau^2/4 < \Delta$, $\Delta > 0$, and $\tau > 0$, we know that $(-1, -3)$ is an *unstable spiral point.*

2) Compute the Laplace Transform of the following function *using the definition of the transform.* Be sure to rewrite your improper integral in terms of a limit, and specify the range of s values where the transform is defined.

$$f(t) = e^{-7t}$$

$$\begin{aligned} F(s) &= \int_0^\infty e^{-7t} e^{-st}\, dt \\ &= \lim_{R\to\infty} \int_0^R e^{-(s+7)t}\, dt \\ &= \lim_{R\to\infty} -\frac{e^{-(s+7)t}}{s+7}\Big|_0^R \\ &= \frac{1}{s+7} - \lim_{R\to\infty} \frac{1}{(s+7)e^{(s+7)t}} \\ &= \frac{1}{s+7} \quad \text{when } s+7>0 \end{aligned}$$

So, $F(s) = \dfrac{1}{s+7}$ on the interval $s > -7$.