Multivariable Calculus
Homework 6 Solutions

Section 12.3 pp. 463, 464

2.) We can take $\vec{r}(x) = \langle x,\ \ln(x),\ 0\rangle$ with the z-component to make taking cross products rigorous. Notice that the domain of the variable is $x > 0$. We first compute the velocity and acceleration vectors.

$$\vec{r}\,'(x) = \left\langle 1,\ \frac{1}{x},\ 0 \right\rangle$$
$$\vec{r}\,''(x) = \left\langle 0,\ -\frac{1}{x^2},\ 0 \right\rangle$$

Notice that the speed of the space curve and the cross product of velocity and acceleration are

$$|\vec{r}\,'(x)| = \sqrt{1 + \frac{1}{x^2}} = \frac{\sqrt{x^2+1}}{x} \quad (x > 0),$$
$$\vec{r}\,'(x) \times \vec{r}\,''(x) = \left\langle 0,\ 0,\ -\frac{1}{x^2} \right\rangle.$$

This gives us

$$\begin{aligned}\kappa(x) &= \frac{|\vec{r}\,'(x) \times \vec{r}\,''(x)|}{|\vec{r}\,'(x)|^3} \\ &= \frac{1/x^2}{(x^2+1)^{3/2}/x^3} \\ &= \frac{x}{(x^2+1)^{3/2}}.\end{aligned}$$

To find where the curvature is largest, we first set $\kappa'(x) = 0$ to identify critical points.

$$\begin{aligned}\kappa'(x) = \frac{1-2x^2}{(x^2+1)^{5/2}} &= 0 \\ 1 - 2x^2 &= 0 \\ x &= \frac{1}{\sqrt{2}} \quad (x > 0).\end{aligned}$$

Notice that $\kappa'(x) > 0$ for $0 < x < 1/\sqrt{2}$ and $\kappa'(x) < 0$ for $x > 1/\sqrt{2}$. As a result, the curvature must have an absolute maximum at $x = 1/\sqrt{2}$.

6.) We take $\vec{r}(t) = \langle \cos^3(t),\ \sin^3(t),\ 0\rangle$. We first compute the velocity and acceleration vectors.

$$\begin{aligned}
\vec{r}\,'(t) &= \big\langle -3\cos^2(t)\sin(t),\ 3\sin^2(t)\cos(t),\ 0\big\rangle \\
\vec{r}\,''(t) &= \big\langle 6\cos(t)\sin^2(t) - 3\cos^3(t),\ 6\sin(t)\cos^2(t) - 3\sin^3(t),\ 0\big\rangle
\end{aligned}$$

Notice that the speed of the space curve and the cross product of velocity and acceleration are

$$\begin{aligned}
|\vec{r}\,'(t)| &= \sqrt{9\cos^4(t)\sin^2(t) + 9\sin^4(t)\cos^2(t)} \\
&= 3|\cos(t)\sin(t)|, \\
\vec{r}\,'(t) \times \vec{r}\,''(t) &= \big\langle 0,\ 0,\ -9\cos^2(t)\sin^2(t)\big\rangle .
\end{aligned}$$

This gives us

$$\begin{aligned}
\kappa(t) &= \frac{|\vec{r}\,'(t) \times \vec{r}\,''(t)|}{|\vec{r}\,'(t)|^3} \\
&= \frac{9\cos^2(t)\sin^2(t)}{27|\cos^3(t)\sin^3(t)|} \\
&= \frac{1}{3|\cos(t)\sin(t)|} \\
&= \frac{2}{3}\left|\frac{1}{\sin(2t)}\right| .
\end{aligned}$$

12.) We take $\vec{r}(t) = \langle v_0\cos(\alpha)t,\ v_0\sin(\alpha)t - gt^2/2,\ 0\rangle$. Since $0 \le \alpha \le \pi/2$, we know that $\cos(\alpha) \ge 0$. We first compute the velocity and acceleration vectors.

$$\begin{aligned}
\vec{r}\,'(t) &= \langle v_0\cos(\alpha),\ v_0\sin(\alpha) - gt,\ 0\rangle \\
\vec{r}\,''(t) &= \langle 0,\ -g,\ 0\rangle
\end{aligned}$$

Notice that the speed of the space curve and the cross product of velocity and acceleration are

$$\begin{aligned}
|\vec{r}\,'(t)| &= \sqrt{v_0^2\cos^2(\alpha) + v_0^2\sin^2(\alpha) - 2gv_0\sin(\alpha)t + g^2t^2} \\
&= \sqrt{v_0^2 - 2gv_0\sin(\alpha)t + g^2t^2}, \\
\vec{r}\,'(t) \times \vec{r}\,''(t) &= \langle 0,\ 0,\ -gv_0\cos(\alpha)\rangle .
\end{aligned}$$

This gives us

$$\begin{aligned}\kappa(t) &= \frac{|\vec{r}\,'(t)\times\vec{r}\,''(t)|}{|\vec{r}\,'(t)|^3}\\ &= \frac{gv_0\cos(\alpha)}{(v_0^2-2gv_0\sin(\alpha)t+g^2t^2)^{3/2}}\end{aligned}$$

25.) For the helix $\vec{r}(t) = \langle\cos(t),\ \sin(t),\ t\rangle$,

$$\begin{aligned}\hat{T}(t) &= \frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|}\\ &= \frac{\langle-\sin(t),\ \cos(t),\ 1\rangle}{\sqrt{\sin^2(t)+\cos^(t)+1}}\\ &= \frac{1}{\sqrt{2}}\langle-\sin(t),\ \cos(t),\ 1\rangle\\ \hat{N}(t) &= \frac{\hat{T}\,'(t)}{|\hat{T}\,'(t)|}\\ &= -\langle\cos(t),\ \sin(t),\ 0\rangle\\ \hat{B}(t) &= \hat{T}(t)\times\hat{N}(t)\\ &= \frac{1}{\sqrt{2}}\langle\sin(t),\ -\cos(t),\ 1\rangle\end{aligned}$$

26.)

$$\begin{aligned}\frac{d}{ds}\left[\hat{B}\cdot\hat{T}\right] &= \frac{d}{ds}[0]\\ \frac{d\hat{B}}{ds}\cdot\hat{T}+\hat{B}\cdot\frac{d\hat{T}}{ds} &= 0\end{aligned}$$

Recall that

$$\frac{d\hat{T}}{ds} = \left|\frac{d\hat{T}}{ds}\right|\frac{\frac{d\hat{T}}{ds}}{\left|\frac{d\hat{T}}{ds}\right|} = \kappa(s)\hat{N},$$

and $\hat{B}$ is perpendicular to $\hat{N}$ (as $\hat{B} = \hat{T}\times\hat{N}$ by definition). This means that $\hat{B}\cdot\hat{T}' = 0$, and the above computation leaves us with

$$\frac{d\hat{B}}{ds}\cdot\hat{T} = 0.$$

As a result, $\hat{B}'$ must be perpendicular to $\hat{T}$. Since $|\hat{B}| = 1$, we know that $\hat{B}'$ must be perpendicular to $\hat{B}$. Since $\hat{B}'$ is perpendicular to both the unit tangent vector and the unit binormal, it must be in the same direction as $\hat{N}$. In other words,

$$\frac{d\hat{B}}{ds} = \alpha\hat{N}$$

for some scalar quantity $\alpha = \alpha(s)$.

By convention, we write

$$\frac{d\hat{B}}{ds} = -\tau(s)\hat{N}$$

where $\tau(s)$ is a quantity known as the *torsion.*

27.) There are two ways to proceed. We could begin by reparametrizing the helix from Problem 25 by arc length. In this case,

$$\begin{aligned} s(t) &= \int_0^t |\vec{r}\,'(u)|\,du \\ &= \int_0^t \sqrt{2}\,du = \sqrt{2}t. \end{aligned}$$

As a result, $t = s/\sqrt{2}$, and using the results of Problem 25 we have

$$\begin{aligned} \hat{B}(s) &= \frac{1}{\sqrt{2}}\left\langle \sin\left(\frac{s}{\sqrt{2}}\right),\ -\cos\left(\frac{s}{\sqrt{2}}\right),\ 1\right\rangle, \\ \frac{d\hat{B}}{ds} &= \frac{1}{2}\left\langle \cos\left(\frac{s}{\sqrt{2}}\right),\ \sin\left(\frac{s}{\sqrt{2}}\right),\ 0\right\rangle, \\ \left|\frac{d\hat{B}}{ds}\right| &= \frac{1}{2}. \end{aligned}$$

So, the helix has a constant torsion of size 1/2.

Since in general it is VERY difficult to reparametrize a curve by arc length, there is another way to proceed. Recall that

$$\frac{ds}{dt} = |\vec{r}\,'(t)|.$$

Using the chain rule, we have

$$\begin{aligned} \frac{d\hat{B}}{ds} &= \frac{dt}{ds}\frac{d\hat{B}}{dt} \\ &= \frac{1}{dt/ds}\frac{d\hat{B}}{dt} \\ &= \frac{1}{|\vec{r}\,'(t)|}\frac{d\hat{B}}{dt}. \end{aligned}$$

This gives us the formula

$$\tau(t) = \frac{|\hat{B}'(t)|}{|\vec{r}\,'(t)|}.$$

Using the results of Problem 25, we get the same answer as above.

$$\tau(t) = \frac{1}{\sqrt{2}}\left|\frac{1}{\sqrt{2}}\langle \cos(t),\ \sin(t),\ 0\rangle\right| = \frac{1}{2}.$$

36.)

$$\begin{aligned}
\vec{r}(t) &= \left\langle e^t\cos(t),\ e^t\sin(t),\ 0\right\rangle \\
\vec{r}\,'(t) &= \left\langle e^t\cos(t) - e^t\sin(t),\ e^t\sin(t) + e^t\cos(t),\ 0\right\rangle \\
\vec{r}\,''(t) &= \left\langle -2e^t\sin(t),\ 2e^t\cos(t),\ 0\right\rangle
\end{aligned}$$

Notice the speed and magnitude of acceleration are given by

$$\begin{aligned}
|\vec{r}\,'(t)| &= e^t\sqrt{(\cos(t)-\sin(t))^2 + (\cos(t)+\sin(t))^2} = \sqrt{2}e^t. \\
|\vec{r}\,''(t)| &= 2e^t
\end{aligned}$$

For the tangential component of acceleration, we have

$$\begin{aligned}
a_T(t) &= \frac{\vec{r}\,''(t)\cdot\vec{r}\,'(t)}{|\vec{r}\,'(t)|} \\
&= \frac{2e^{2t}}{\sqrt{2}e^t} = \sqrt{2}e^t.
\end{aligned}$$

For the normal component of acceleration, we have

$$\begin{aligned}
a_N(t) &= \sqrt{|\vec{r}\,''(t)|^2 - (a_t(t))^2} \\
&= \sqrt{4e^{2t} - 2e^{2t}} = \sqrt{2}e^t.
\end{aligned}$$

38.)

$$\vec{r}(t)\cdot\vec{r}\,''(t) = 2e^{2t}\left(-\sin(t)\cos(t) + \sin(t)\cos(t)\right) = 0$$

So, the position and acceleration vectors are always at a right angle to each other.