Multivariable Calculus
Homework 5 Solutions

Section 12.1 pp. 452, 453

6.)

(a) $x = 2^t, y = 4^t$

$$\begin{aligned} y &= 2^{2t} \\ y &= \left(2^t\right)^2 \\ y &= x^2 \qquad (x > 0) \end{aligned}$$

(b) $x = 4^t, y = 8^t$

$$\begin{aligned} y &= \left(2^t\right)^3 \\ x &= \left(2^t\right)^2 \longrightarrow \sqrt{x} = 2^t \\ y &= x^{3/2} \qquad (x > 0) \end{aligned}$$

(c) $x = 4^t, y = 4t$

$$\begin{aligned} x &= 4^t \longrightarrow t = \log_4(x) \\ y &= 4\log_4(x) = \frac{4\ln(x)}{\ln(4)} \end{aligned}$$

9.)

(a) $\vec{R}(t) = \langle t + 2,\ 3,\ -t + 4 \rangle$

(b) $\vec{R}(t) = \left\langle \frac{t^2}{2} + 2,\ 3,\ -\frac{t^2}{2} + 4 \right\rangle$

(c) The path is still a line.

20.) If $\vec{r}(t) = \langle x(t),\ y(t)\rangle$, then

$$\vec{r}(t)\cdot\vec{r}\,'(t) = x(t)x'(t) + y(t)y'(t) = \frac{1}{2}\frac{d}{dt}\left[\vec{r}(t)\cdot\vec{r}(t)\right].$$

If this quantity is zero, then $\vec{r}(t)\cdot\vec{r}(t) = R^2$ for some positive constant R. But this means that

$$|\vec{r}(t)| = R$$

and the point travels on a circle of radius of R for all time.

23.)

$$\begin{aligned}\vec{r}\,'(t) &= \langle 1-\sin(t),\ 1-\cos(t)\rangle\\ |\vec{r}\,'(t)| &= \sqrt{(1-\sin(t))^2 + (1-\cos(t))^2} = \sqrt{3 - 2\sin(t) - 2\cos(t)}\end{aligned}$$

The speed is a maximum/minimum when the quantity $3 - 2\sin(t) - 2\cos(t)$ is a maximum/minimum. Setting the derivative of this quantity equal to zero gives critical points of $t = \pi/4$ and $5\pi/4$ (and any angle coterminal with these). The maximum is clearly at $t = 5\pi/4$ (where both sine and cosine are negative) giving

$$|\vec{r}\,'(5\pi/4)| = \sqrt{3 + 2\sqrt{2}}.$$

Notice the minimum value of the speed is $\sqrt{3 - 2\sqrt{2}}$ at $t = \pi/4$.

For the acceleration:

$$\begin{aligned}\vec{r}\,''(t) &= \langle -\cos(t),\ \sin(t)\rangle,\\ |\vec{r}\,''(t)| &= 1.\end{aligned}$$

Also, notice that the point is moving around the circle of radius 1 whose center is at the point (t, t) on the line $y = x$.

34.) One possible answer is

$$\vec{r}(t) = \left\langle \cos(e^{-t}),\ \sin(e^{-t})\right\rangle.$$

Notice that $x(t)^2 + y(t)^2 = \cos^2(e^{-t}) + \sin^2(e^{-t}) = 1$ so that the particle definitely moves on the unit circle. For the speed:

$$\begin{aligned}\vec{r}\,'(t) &= -e^{-t}\left\langle -\sin(e^{-t}),\ \cos(e^{-t})\right\rangle,\\ |\vec{r}\,'(t)| &= e^{-t}.\end{aligned}$$

If the author means "Do we make it all the around starting at $t = 0$," then the answer is clearly no. For $0 \le t < \infty$, e^{-t} starts out at 1 and limits to 0. If we allow *negative* times, then the answer is clearly yes!

Section 12.2 pp. 457, 458

1.) Since the projectile starts out from the ground with an initial speed of 16 ft/sec, we have

$$\vec{r}(t) = \left\langle 16\cos(\alpha)t,\ -\frac{1}{2}gt^2 + 16\sin(\alpha)t \right\rangle.$$

The time of flight is $T > 0$ such that $y(T) = 0$. Clearly,

$$T = \frac{32\sin(\alpha)}{g}.$$

This gives the range, R, as

$$R = x\left(\frac{32\sin(\alpha)}{g}\right) = \frac{256\sin(2\alpha)}{g}.$$

In general, the maximum height, Y, is found by solving $y'(t) = 0$. In this problem, symmetry forces the maximum height to occur at $T/2$. This gives

$$Y = y\left(\frac{16\sin(\alpha)}{g}\right) = \frac{128\sin^2(\alpha)}{g}.$$

In the answers below, we use $g = 32$ ft/sec^2.

(a) $\alpha = 30°$

$$T = \frac{1}{2} \text{ sec}$$
$$R = 4\sqrt{3} \text{ ft}$$
$$Y = 1 \text{ ft}$$

(b) $\alpha = 60°$

$$T = \frac{\sqrt{3}}{2} \text{ sec}$$
$$R = 4\sqrt{3} \text{ ft}$$
$$Y = 3 \text{ ft}$$

(c) $\alpha = 90°$

$$T = 1 \text{ sec}$$
$$R = 0 \text{ ft}$$
$$Y = 4 \text{ ft}$$

6.) I believe the author means that the ball lands 6 meters away when it is kicked with an initial speed of 30 m/sec. So, we need to find the angle α so that

$$R = \frac{30^2 \sin(2\alpha)}{g} = 6.$$

That means we need

$$\sin(2\alpha) = \frac{6g}{900}$$

where $g = 9.8$ m/sec^2. Since $0 \le \alpha \le \pi/2$, there are two angles that will do the job:

$$\alpha = \frac{1}{2}\arcsin\left(\frac{6g}{900}\right),\ \frac{\pi}{2} - \frac{1}{2}\arcsin\left(\frac{6g}{900}\right).$$

These angles work out to $\alpha \approx 1.873°,\ 88.127°$.

16.) The speed of a projectile following the path

$$\vec{r}(t) = \left\langle v_0 \cos(\theta)t,\ -\frac{1}{2}gt^2 + v_0 \sin(\theta)t \right\rangle,$$

is given by

$$|\vec{r}\,'(t)| = \sqrt{v_0^2 - 2v_0 gt\sin(\theta) + (gt)^2}.$$

This quantity is a maximum whenever $v_0^2 - 2v_0 gt\sin(\theta) + (gt)^2$ is maximized over the time interval

$$0 \le t \le \frac{2v_0 \sin(\alpha)}{g}.$$

Taking the derivative and setting this equal to zero gives

$$t = \frac{v_0 \sin(\theta)}{g}$$

as the only critical point. Notice that this is precisely the top of its trajectory (i.e. the time when it is highest off the ground).

Time	Speed
0	v_0
$\frac{v_0 \sin(\theta)}{g}$	$v_0 \cos(\theta)$
$\frac{2v_0 \sin(\theta)}{g}$	v_0

Clearly, the projectile is going the slowest at the top of its path and fastest when it is launched and when it lands.

19.)

$$\begin{aligned}
\vec{r}(t) &= R\langle \theta(t) - \sin\theta(t),\ 1 - \cos\theta(t)\rangle \\
\vec{r}\,'(t) &= R\theta'(t)\langle 1 - \cos\theta(t),\ \sin\theta(t)\rangle \\
|\vec{r}\,'(t)| &= \sqrt{2}R\theta'(t)\sqrt{1-\cos\theta(t)} \qquad (\text{ assuming } \theta'(t) \geq 0) \\
\hat{T}(t) &= \frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|} = \frac{\langle 1 - \cos\theta(t),\ \sin\theta(t)\rangle}{\sqrt{2}\sqrt{1-\cos\theta(t)}}
\end{aligned}$$

For $\theta = 0$ and $\theta'(t) = 1$, we have

$$|\vec{r}\,'(0)| = 0.$$

Similarly, when $\theta = \pi$ and $\theta'(t) = 1$, we have

$$|\vec{r}\,'(\pi)| = 2R.$$

25.) If $\theta'(t) = c$ (a positive constant) and $\theta(0) = 0$, then we can take $\theta(t) = ct$. This gives

$$\begin{aligned}
\vec{r}(t) &= R\langle ct - \sin(ct),\ 1 - \cos(ct)\rangle, \\
\vec{r}\,'(t) &= cR\langle 1 - \cos(ct),\ \sin(ct)\rangle, \\
|\vec{r}\,'(t)| &= \sqrt{2}cR\sqrt{1-\cos(ct)}.
\end{aligned}$$

Clearly, dx/dt is greatest whenever $ct = \pi, 3\pi, 5\pi, 7\pi, \cdots$ when the horizontal velocity is $2cR$ (notice the the horizontal velocity is 0 when $ct = 0, 2\pi, 4\pi, 6\pi, \cdots$). Likewise, dy/dt is greatest when $ct = \pi/2, 5\pi/2, 9\pi/2, \cdots$ when it equals cR (it achieves a minimum of $-cR$ at $ct = 3\pi/2, 7\pi/2, 11\pi/2, \cdots$).