Multivariable Calculus
Homework 4 Solutions

Section 8.9 p. 436

18.)

Eigenvalues:

$$\begin{aligned}\begin{vmatrix}-1-\lambda & 1\\ 3 & 1-\lambda\end{vmatrix} &= (-1-\lambda)(1-\lambda)-3\\ &= \lambda^2-4=0\\ \lambda &= -2, 2\end{aligned}$$

Eigenvectors:

- $\lambda = -2$

$$\left[\begin{array}{cc|c}1 & 1 & 0\\ 3 & 3 & 0\end{array}\right] \sim \left[\begin{array}{cc|c}1 & 1 & 0\\ 0 & 0 & 0\end{array}\right]$$

So, the family of eigenvectors is

$$c\begin{bmatrix}1\\-1\end{bmatrix}.$$

- $\lambda = 2$

$$\left[\begin{array}{cc|c}-3 & 1 & 0\\ 3 & -1 & 0\end{array}\right] \sim \left[\begin{array}{cc|c}3 & -1 & 0\\ 0 & 0 & 0\end{array}\right]$$

So, the family of eigenvectors is

$$c\begin{bmatrix}1\\3\end{bmatrix}.$$

20.)

Eigenvalues:

$$\begin{aligned}\begin{vmatrix}7-\lambda & -13\\ 1 & 1-\lambda\end{vmatrix} &= (7-\lambda)(1-\lambda)+13\\ &= \lambda^2-8\lambda+20=0\\ \lambda &= 4\pm 2i\end{aligned}$$

Eigenvectors:

- $\lambda = 4 + 2i$

$$\left[\begin{array}{cc|c} 7-(4+2i) & -13 & 0 \\ 1 & 1-(4+2i) & 0 \end{array}\right] = \left[\begin{array}{cc|c} 3-2i & -13 & 0 \\ 1 & -3-2i & 0 \end{array}\right]$$
$$\sim \left[\begin{array}{cc|c} 13 & -13(3+2i) & 0 \\ 1 & -(3+2i) & 0 \end{array}\right]$$
$$\sim \left[\begin{array}{cc|c} 1 & -(3+2i) & 0 \\ 0 & 0 & 0 \end{array}\right]$$

So, the family of eigenvectors is

$$c\begin{bmatrix} 3+2i \\ 1 \end{bmatrix}.$$

- $\lambda = 4 - 2i$

The family of eigenvectors is

$$c\begin{bmatrix} 3-2i \\ 1 \end{bmatrix}.$$

22.)

Eigenvalues:

$$\begin{vmatrix} 3-\lambda & 1 & -1 \\ 1 & 3-\lambda & -1 \\ 1 & 1 & 1-\lambda \end{vmatrix} = (3-\lambda)\begin{vmatrix} 3-\lambda & -1 \\ 1 & 1-\lambda \end{vmatrix} - \begin{vmatrix} 1 & -1 \\ 1 & 1-\lambda \end{vmatrix} - \begin{vmatrix} 1 & 3-\lambda \\ 1 & 1 \end{vmatrix}$$
$$= (3-\lambda)\Big((3-\lambda)(1-\lambda)+1\Big) - \Big(1-\lambda+1\Big) - \Big(1-(3-\lambda)\Big)$$
$$= (3-\lambda)\Big(\lambda^2-4\lambda+4\Big) - \Big(2-\lambda\Big) - \Big(\lambda-2\Big)$$
$$= (3-\lambda)(\lambda-2)^2 = 0$$
$$\lambda = 2, 2, 3$$

Eigenvectors:

- $\lambda = 2$

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 1 & 1 & -1 & 0 \\ 1 & 1 & -1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Since there are two rows of zeros, we actually get two independent families of eigenvectors. The easiest way to select them is by choosing $v_2 = 0$ for the first family and $v_3 = 0$ for the second. That gives the family of eigenvectors as

$$c_1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}.$$

- $\lambda = 3$

$$\left[\begin{array}{ccc|c} 0 & 1 & -1 & 0 \\ 1 & 0 & -1 & 0 \\ 1 & 1 & -2 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & 1 & -2 & 0 \end{array}\right]$$
$$\sim \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right]$$
$$\sim \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

So, the family of eigenvectors is

$$c \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

24.)

Eigenvalues:

$$\begin{vmatrix} -10-\lambda & -6 & 28 \\ 6 & 5-\lambda & -19 \\ -2 & -1 & 5-\lambda \end{vmatrix} = (-10-\lambda) \begin{vmatrix} 5-\lambda & -19 \\ -1 & 5-\lambda \end{vmatrix} + 6 \begin{vmatrix} 6 & -19 \\ -2 & 5-\lambda \end{vmatrix} + 28 \begin{vmatrix} 6 & 5-\lambda \\ -2 & -1 \end{vmatrix}$$
$$= (-10-\lambda)\Big((5-\lambda)^2 - 19\Big) + 6\Big(6(5-\lambda) - 38\Big) + 28\Big(-6 + 2(5-\lambda)\Big)$$
$$= -\Big(\lambda^3 - 2\lambda - 4\Big)$$
$$= -(\lambda - 2)(\lambda^2 + 2\lambda + 2) = 0$$
$$\lambda = 2, -1 \pm i$$

Eigenvectors:

- $\lambda = 2$

$$\left[\begin{array}{ccc|c} -12 & -6 & 28 & 0 \\ 6 & 3 & -19 & 0 \\ -2 & -1 & 3 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 6 & 3 & -14 & 0 \\ 6 & 3 & -19 & 0 \\ 2 & 1 & -3 & 0 \end{array}\right]$$
$$\sim \left[\begin{array}{ccc|c} 6 & 3 & -14 & 0 \\ 0 & 0 & -5 & 0 \\ 2 & 1 & -3 & 0 \end{array}\right]$$
$$\sim \left[\begin{array}{ccc|c} 6 & 3 & -14 & 0 \\ 2 & 1 & -3 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$
$$\sim \left[\begin{array}{ccc|c} 6 & 3 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$
$$\sim \left[\begin{array}{ccc|c} 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

So, the family of eigenvectors is

$$c \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}.$$

- $\lambda = -1 + i$

$$\left[\begin{array}{ccc|c} -9-i & -6 & 28 & 0 \\ 6 & 6-i & -19 & 0 \\ -2 & -1 & 6-i & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 82 & -6(-9+i) & 28(-9+i) & 0 \\ 6 & 6-i & -19 & 0 \\ 2 & 1 & -6+i & 0 \end{array}\right]$$
$$\sim \left[\begin{array}{ccc|c} 2 & 1 & -6+i & 0 \\ 6 & 6-i & -19 & 0 \\ 82 & 54-6i & -252+28i & 0 \end{array}\right]$$
$$\sim \left[\begin{array}{ccc|c} 2 & 1 & -6+i & 0 \\ 0 & 3-i & -1-3i & 0 \\ 0 & 13-6i & -6-13i & 0 \end{array}\right]$$
$$\sim \left[\begin{array}{ccc|c} 2 & 1 & -6+i & 0 \\ 0 & 10 & -10i & 0 \\ 0 & 205 & -205i & 0 \end{array}\right]$$
$$\sim \left[\begin{array}{ccc|c} 2 & 1 & -6+i & 0 \\ 0 & 1 & -i & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$
$$\sim \left[\begin{array}{ccc|c} 1 & 0 & -(3-i) & 0 \\ 0 & 1 & -i & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

So, the family of eigenvectors is

$$c\begin{bmatrix}3-i\\ i\\ 1\end{bmatrix}.$$

- $\lambda = -1-i$

  The family of eigenvectors is

$$c\begin{bmatrix}3+i\\ -i\\ 1\end{bmatrix}.$$

26.)

Eigenvalues: Notice that it is easiest here to expand on the last row! Moving this row to the top of the determinant involves 3 row swaps. This officially introduces a minus sign into the resulting determinant (though this will not impact the eigenvalues).

$$\begin{aligned}\begin{vmatrix}-1-\lambda & 4 & -2 & -4\\ -2 & 2-\lambda & 2 & 1\\ -2 & -1 & -1-\lambda & 1\\ 0 & 3 & 0 & -\lambda\end{vmatrix} &= -\begin{vmatrix}0 & 3 & 0 & -\lambda\\ -1-\lambda & 4 & -2 & -4\\ -2 & 2-\lambda & 2 & 1\\ -2 & -1 & -1-\lambda & 1\end{vmatrix}\\ &= 3\begin{vmatrix}-1-\lambda & -2 & -4\\ -2 & 2 & 1\\ -2 & -1-\lambda & 1\end{vmatrix} - \lambda\begin{vmatrix}-1-\lambda & 4 & -2\\ -2 & 2-\lambda & 2\\ -2 & -1 & -1-\lambda\end{vmatrix}\\ &= 3\Big(-\lambda^2-12\lambda-27\Big) - \lambda\Big(-\lambda^3-3\lambda-36\Big)\\ &= \lambda^4-81 = (\lambda^2-9)(\lambda^2+9)\\ \lambda &= -3, 3, -3i, 3i\end{aligned}$$

Eigenvectors:

- $\lambda = -3$

$$\left[\begin{array}{cccc|c} 2 & 4 & -2 & -4 & 0 \\ -2 & 5 & 2 & 1 & 0 \\ -2 & -1 & 2 & 1 & 0 \\ 0 & 3 & 0 & 3 & 0 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 2 & -1 & -2 & 0 \\ -2 & 5 & 2 & 1 & 0 \\ -2 & -1 & 2 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \end{array}\right]$$
$$\sim \left[\begin{array}{cccc|c} 1 & 2 & -1 & -2 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 3 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$
$$\sim \left[\begin{array}{cccc|c} 1 & 0 & -1 & -4 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$
$$\sim \left[\begin{array}{cccc|c} 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

So, the family of eigenvectors is

$$c \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}.$$

- $\lambda = 3$

$$\left[\begin{array}{cccc|c} -4 & 4 & -2 & -4 & 0 \\ -2 & -1 & 2 & 1 & 0 \\ -2 & -1 & -4 & 1 & 0 \\ 0 & 3 & 0 & -3 & 0 \end{array}\right] \sim \left[\begin{array}{cccc|c} 2 & -2 & 1 & 2 & 0 \\ -2 & -1 & 2 & 1 & 0 \\ -2 & -1 & -4 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 \end{array}\right]$$
$$\sim \left[\begin{array}{cccc|c} 2 & -2 & 1 & 2 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & -1 & 0 \\ 0 & 1 & 1 & -1 & 0 \end{array}\right]$$
$$\sim \left[\begin{array}{cccc|c} 2 & -2 & 0 & 2 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{array}\right]$$
$$\sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

So, the family of eigenvectors is

$$c\begin{bmatrix}0\\1\\0\\1\end{bmatrix}.$$

- $\lambda = 3i$

$$\left[\begin{array}{cccc|c}-1-3i & 4 & -2 & -4 & 0\\ -2 & 2-3i & 2 & 1 & 0\\ -2 & -1 & -1-3i & 1 & 0\\ 0 & 3 & 0 & -3i & 0\end{array}\right] \sim \left[\begin{array}{cccc|c}10 & 4(-1+3i) & -2(-1+3i) & -4(-1+3i) & 0\\ 0 & 1 & 0 & -i & 0\\ -2 & 2-3i & 2 & 1 & 0\\ -2 & -1 & -1-3i & 1 & 0\end{array}\right]$$

$$\sim \left[\begin{array}{cccc|c}5 & -2+6i & 1-3i & 2-6i & 0\\ 0 & 1 & 0 & -i & 0\\ -2 & 2-3i & 2 & 1 & 0\\ -2 & -1 & -1-3i & 1 & 0\end{array}\right]$$

$$\sim \left[\begin{array}{cccc|c}1 & -4+6i & -1-9i & 4-6i & 0\\ 0 & 1 & 0 & -i & 0\\ -2 & 2-3i & 2 & 1 & 0\\ -2 & -1 & -1-3i & 1 & 0\end{array}\right]$$

$$\sim \left[\begin{array}{cccc|c}1 & -4+6i & -1-9i & 4-6i & 0\\ 0 & 1 & 0 & -i & 0\\ 0 & -2+3i & -6i & 3-4i & 0\\ 0 & -3+4i & -1-7i & 3-4i & 0\end{array}\right]$$

$$\sim \left[\begin{array}{cccc|c}1 & 0 & -1-9i & -2-10i & 0\\ 0 & 1 & 0 & -i & 0\\ 0 & 0 & 1 & 1 & 0\\ 0 & 0 & 0 & 0 & 0\end{array}\right]$$

$$\sim \left[\begin{array}{cccc|c}1 & 0 & 0 & -(1+i) & 0\\ 0 & 1 & 0 & -i & 0\\ 0 & 0 & 1 & 1 & 0\\ 0 & 0 & 0 & 0 & 0\end{array}\right]$$

So, the family of eigenvectors is

$$c\begin{bmatrix}1+i\\i\\-1\\1\end{bmatrix}.$$

- $\lambda = -3i$ The family of eigenvectors is

$$c\begin{bmatrix}1-i\\-i\\-1\\1\end{bmatrix}.$$