Multivariable Calculus
Homework 2 Solutions

Section 11.2 pp. 414, 415

2.) A normal vector for the plane is $\vec{N} = \langle 1, 2, 3 \rangle$. While there are an infinite number of points on the plane, two easy ones are $P = (6, 0, 0)$ and $P_0 = (0, 3, 0)$. Using these points, we have

$$\begin{aligned} \vec{N} \cdot (P - P_0) &= \langle 1, 2, 3 \rangle \cdot \langle 6, -3, 0 \rangle \\ &= 1(6) + 2(-3) + 3(0) = 0. \end{aligned}$$

8.)

$$\begin{aligned} \vec{N} \cdot (P - P_0) &= 0 \\ \langle 1, 2, -1 \rangle \cdot \langle x - 1, y - 2, z + 1 \rangle &= 0 \\ 1(x - 1) + 2(y - 2) - (z + 1) &= 0 \\ x + 2y - z &= 6 \end{aligned}$$

10.) The given plane has a normal vector of $\vec{N} = \langle 1, 1, 1 \rangle$, and so the plane we are looking for must have a normal vector that is a multiple of this one. So, we can just take the same vector!

$$\begin{aligned} \vec{N} \cdot (P - P_0) &= 0 \\ \langle 1, 1, 1 \rangle \cdot \langle x - x_0, y - y_0, z - z_0 \rangle &= 0 \\ (x - x_0) + (y - y_0) + (z - z_0) &= 0 \\ x + y + z &= x_0 + y_0 + z_0 \end{aligned}$$

20.)

$$\begin{aligned}
P_{\vec{A}}\vec{B} &= \left(\frac{\vec{A}\cdot\vec{B}}{|\vec{A}|}\right)\ \frac{\vec{A}}{|\vec{A}|} \\
&= \left(\frac{\langle 1,-1,0\rangle\cdot\langle 4,2,4\rangle}{|\langle 1,-1,0\rangle|}\right)\ \frac{\langle 1,-1,0\rangle}{|\langle 1,-1,0\rangle|} \\
&= \left(\frac{2}{\sqrt{2}}\right)\ \frac{\langle 1,-1,0\rangle}{\sqrt{2}} \\
&= \langle 1,-1,0\rangle
\end{aligned}$$

$$|P_{\vec{A}}\vec{B}| = \sqrt{2}$$

34.) Recall that we can find the distance from a point Q_0 to a point P_0 on a plane with normal vector $\vec{N}$ by forming the vector $\overrightarrow{P_0Q_0}$ and computing the magnitude of the projection of this vector on the normal vector.

$$\begin{aligned}
\left|P_{\vec{N}}\overrightarrow{P_0Q_0}\right| &= \frac{\left|\vec{N}\cdot\overrightarrow{P_0Q_0}\right|}{|\vec{N}|} \\
&= \frac{|\langle 1,2,2\rangle\cdot\langle -1,-1,-1\rangle|}{|\langle 1,2,2\rangle|} \\
&= \frac{|-1-2-2|}{\sqrt{9}} \\
&= \frac{5}{3}
\end{aligned}$$

Section 11.3 pp. 423 – 425

4.)

$$\begin{aligned}\langle 2,3,1\rangle \times \langle 2,3,-1\rangle &= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & 3 & 1 \\ 2 & 3 & -1 \end{vmatrix} \\ &= (-3-3)\,\hat{\imath} - (-2-2)\,\hat{\jmath} + (6-6)\,\hat{k} \\ &= \langle -6, 4, 0\rangle\end{aligned}$$

6.)

$$\begin{aligned}\langle 1,1,-1\rangle \times \langle 1,-1,1\rangle &= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 1 & -1 \\ 1 & -1 & 1 \end{vmatrix} \\ &= (1-1)\,\hat{\imath} - (1+1)\,\hat{\jmath} + (-1-1)\,\hat{k} \\ &= \langle 0, -2, -2\rangle\end{aligned}$$

10.)

(a) **TRUE**: $\vec{A} \times \vec{B}$ is a vector while $\vec{A} \cdot \vec{B}$ is a scalar!

(b) **TRUE**: If $\vec{A} \times \vec{B} = \vec{0}$, then these two vectors are parallel or one of them is the zero vector. If $\vec{A} = \vec{0}$, then the statement is true. Otherwise, $\vec{B} = \lambda\vec{A}$ where λ is some scalar quantity (potentially equal to zero). This means

$$\begin{aligned}\vec{A} \cdot \vec{B} &= \vec{A} \cdot (\lambda\vec{A}) \\ &= \lambda|\vec{A}|^2 = 0.\end{aligned}$$

Since we are assuming $\vec{A} \neq \vec{0}$, we must have $\lambda = 0$. But then $\vec{B} = \vec{0}$. In either case, the given statement is true.

(c) **FALSE:**

$$\begin{aligned}\vec{A} \times \vec{B} &= \vec{A} \times \vec{C} \\ \vec{A} \times \left(\vec{B} - \vec{C}\right) &= \vec{0}\end{aligned}$$

This means that $\vec{B} - \vec{C}$ must be the zero vector or parallel to $\vec{A}$. So generally, we have

$$\begin{aligned}\vec{B} - \vec{C} &= \lambda\vec{A} \\ \vec{B} &= \vec{C} + \lambda\vec{A}\end{aligned}$$

where λ is some scalar. So pick any non zero vectors $\vec{A}$ and $\vec{C}$ and define $\vec{B} = \vec{C} + \vec{A}$ for a counter example.

24.) Let $P = (0, 1, 2), Q = (1, 2, 3)$, and $R = (2, 3, 4)$. To get a normal vector, we can take

$$\vec{N} = \overrightarrow{PQ} \times \overrightarrow{PR}$$

or any convenient (non-zero) multiple of this vector.

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 1 & 1 \\ 2 & 2 & 2 \end{vmatrix} = \vec{0}.$$

This means that the three points lie on the same line which is given by

$$\vec{r}(t) = \langle t, t+1, t+2 \rangle.$$

As such, there are an *infinite* number of planes containing these three points!

36.)

(a)

$$\begin{aligned}(\vec{A} + \vec{B}) \times \vec{B} &= \vec{A} \times \vec{B} + \vec{B} \times \vec{B} \\ &= \vec{A} \times \vec{B} + \vec{0} = \vec{A} \times \vec{B}\end{aligned}$$

(b)

$$\begin{aligned}(-\vec{B}) \times (-\vec{A}) &= (-1)^2(\vec{B} \times \vec{A}) \\ &= \vec{B} \times \vec{A} = -(\vec{A} \times \vec{B})\end{aligned}$$

(c) $|\vec{A}||\vec{B}||\sin(\theta)|$ cannot be equal to $\vec{A} \times \vec{B}$ since the first is a scalar while the second is a vector. However

$$|\vec{A} \times \vec{B}| = |\vec{A}||\vec{B}|\sin(\theta)$$

where θ is the angle between the two vectors.

(d)

$$\begin{aligned}(\vec{A} + \vec{C}) \times (\vec{B} - \vec{C}) &= \vec{A} \times \vec{B} - \vec{A} \times \vec{C} + \vec{C} \times \vec{B} - \vec{C} \times \vec{C} \\ &= \vec{A} \times \vec{B} - \vec{A} \times \vec{C} - \vec{B} \times \vec{C}\end{aligned}$$

(e)

$$\begin{aligned}\frac{1}{2}(\vec{A} - \vec{B}) \times (\vec{A} + \vec{B}) &= \frac{1}{2}\left(\vec{A} \times \vec{A} + \vec{A} \times \vec{B} - \vec{B} \times \vec{A} - \vec{B} \times \vec{B}\right) \\ &= \frac{1}{2}\left(\vec{A} \times \vec{B} - \vec{B} \times \vec{A}\right) \\ &= \frac{1}{2}\left(\vec{A} \times \vec{B} + \vec{A} \times \vec{B}\right) \\ &= \vec{A} \times \vec{B}\end{aligned}$$

42.) Let $P = (a_1, b_1, 0), Q = (a_2, b_2, 0)$, and $R = (a_3, b_3, 0)$. We've put these points in $\mathbb{R}^3$ so that we can use the cross product. Two of the sides of the resulting triangle are

$$\overrightarrow{PQ} = \langle a_2 - a_1, b_2 - b_1, 0\rangle,$$
$$\overrightarrow{PR} = \langle a_3 - a_1, b_3 - b_1, 0\rangle.$$

The cross product of these two vectors is

$$\begin{aligned} \overrightarrow{PQ} \times \overrightarrow{PR} &= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ a_2 - a_1 & b_2 - b_1 & 0 \\ a_3 - a_1 & b_3 - b_1 & 0 \end{vmatrix} \\ &= ((a_2 - a_1)(b_3 - b_1) - (a_3 - a_1)(b_2 - b_1))\,\hat{k}. \end{aligned}$$

Since the area of the triangle is half the area of the parallelogram with sides $\overrightarrow{PQ}$ and $\overrightarrow{PR}$, we have

$$\begin{aligned} \text{Area}(\triangle PQR) &= \frac{1}{2}|\overrightarrow{PQ} \times \overrightarrow{PR}| \\ &= \frac{1}{2}|(a_2 - a_1)(b_3 - b_1) - (a_3 - a_1)(b_2 - b_1)|. \end{aligned}$$

47.)

(a) $(\vec{A} \times \vec{B}) \times \vec{C}$ is in the plane containing both $\vec{A}$ and $\vec{B}$ because it is perpendicular to $\vec{A} \times \vec{B}$ which is a normal vector to that plane. This means that

$$(\vec{A} \times \vec{B}) \times \vec{C} = \lambda_1\vec{A} + \lambda_2\vec{B}$$

for some scalars λ_1 and λ_2.

(b)

$$\begin{aligned} \vec{A} \times \vec{B} &= \langle a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1\rangle \\ (\vec{A} \times \vec{B}) \times \vec{C} &= (\vec{A} \times \vec{B}) \times \hat{\imath} \\ &= (a_2b_3 - a_3b_2)(\hat{\imath} \times \hat{\imath}) + (a_3b_1 - a_1b_3)(\hat{\jmath} \times \hat{\imath}) + (a_1b_2 - a_2b_1)(\hat{k} \times \hat{\imath}) \\ &= -(a_3b_1 - a_1b_3)\hat{k} + (a_1b_2 - a_2b_1)\hat{\jmath} \\ &= \langle 0, a_1b_2 - a_2b_1, a_1b_3 - a_3b_1\rangle \end{aligned}$$

(c)

$$\begin{aligned} (\vec{A} \cdot \vec{C})\vec{B} - (\vec{B} \cdot \vec{C})\vec{A} &= (\vec{A} \cdot \hat{\imath})\vec{B} - (\vec{B} \cdot \hat{\imath})\vec{A} \\ &= a_1\langle b_1, b_2, b_3\rangle - b_1\langle a_1, a_2, a_3\rangle \\ &= \langle 0, a_1b_2 - a_2b_1, a_1b_3 - a_3b_1\rangle \\ &= (\vec{A} \times \vec{B}) \times \vec{C} \end{aligned}$$