Multivariable Calculus
Homework 1 Solutions

2.)

$$\vec{V} + \vec{W} = \hat{\imath} + 2\hat{\jmath} - \hat{k}$$
$$2\vec{V} - 3\vec{W} = 2\hat{\imath} - \hat{\jmath} + 3\hat{k}$$
$$|\vec{V}|^2 = \vec{V} \cdot \vec{V} = 1 + 1 = 2$$
$$\vec{V} - \vec{W} = \hat{\imath} + \hat{k}$$
$$\cos(\theta) = \frac{\vec{V} \cdot \vec{W}}{|\vec{V}||\vec{W}|} = \frac{1}{(\sqrt{2})(\sqrt{2})} = \frac{1}{2}$$

3.)

$$\vec{V} + \vec{W} = 2\hat{\imath} - \hat{\jmath} - \hat{k}$$
$$2\vec{V} - 3\vec{W} = -\hat{\imath} - 7\hat{\jmath} + 8\hat{k}$$
$$|\vec{V}|^2 = \vec{V} \cdot \vec{V} = 1 + 4 + 1 = 6$$
$$\vec{V} - \vec{W} = -3\hat{\jmath} + 3\hat{k}$$
$$\cos(\theta) = \frac{\vec{V} \cdot \vec{W}}{|\vec{V}||\vec{W}|} = \frac{1 - 2 - 2}{(\sqrt{6})(\sqrt{6})} = -\frac{1}{2}$$

6.) There are an infinite number of possible answers here. One vector that is clearly perpendicular to $\vec{v} = \langle 1, 1, 0 \rangle$ is $\hat{k} = \langle 0, 0, 1 \rangle$. The second vector must be perpendicular to $\hat{k}$ and must be in the xy-plane. One particular vector in the xy-plane that is perpendicular to the given vector $\vec{v}$ is $\vec{v}_1 = \langle 1, -1, 0 \rangle$.

13.) This problem can easily be solved graphically. We can also do it algebraically as follows. We will use $\vec{v}_n$ to denote the vector pointing to the number n and that each vector has unit length. Each number on the edge of the clock is $30°$ from the next.

$\vec{v}_{12} = \langle 0, 1 \rangle$	$\vec{v}_6 = \langle 0, -1 \rangle$	$\vec{v}_3 = \langle 1, 0 \rangle$	$\vec{v}_9 = \langle -1, 0 \rangle$
$\vec{v}_1 = \left\langle \frac{1}{2}, \frac{\sqrt{3}}{2} \right\rangle$	$\vec{v}_7 = \left\langle -\frac{1}{2}, -\frac{\sqrt{3}}{2} \right\rangle$	$\vec{v}_4 = \left\langle \frac{\sqrt{3}}{2}, -\frac{1}{2} \right\rangle$	$\vec{v}_{10} = \left\langle -\frac{\sqrt{3}}{2}, \frac{1}{2} \right\rangle$
$\vec{v}_2 = \left\langle \frac{\sqrt{3}}{2}, \frac{1}{2} \right\rangle$	$\vec{v}_8 = \left\langle -\frac{\sqrt{3}}{2}, -\frac{1}{2} \right\rangle$	$\vec{v}_5 = \left\langle \frac{1}{2}, -\frac{\sqrt{3}}{2} \right\rangle$	$\vec{v}_{11} = \left\langle -\frac{1}{2}, \frac{\sqrt{3}}{2} \right\rangle$

(a) As you can see from the table above, all of the vectors cancel out leaving the sum to be the zero vector.

(b) If we remove $\vec{v}_4$ and sum up the remaining vectors, the sum is $\vec{v}_{10}$, the vector pointing to 10 O'Clock.

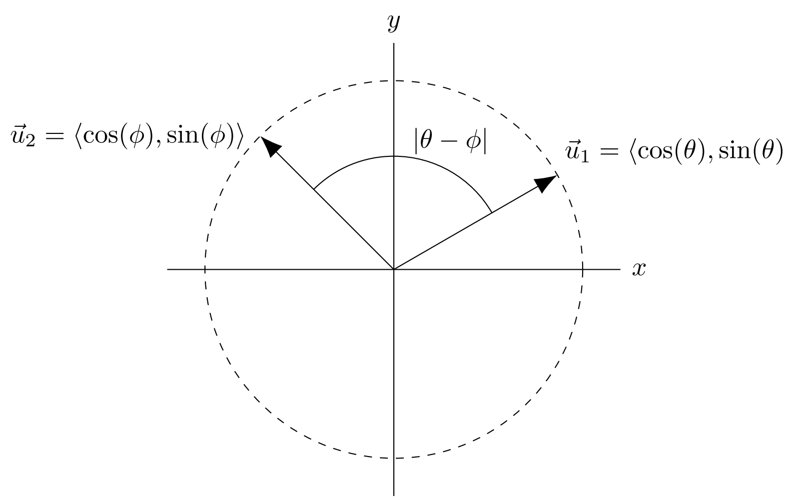
(c) If we reduce the length of the vectors $\vec{v}_1, \vec{v}_2$, and $\vec{v}_3$ by half, the sum of the 12 vectors leaves us with a multiple of the vector pointing to 8 O'Clock.

$$\begin{aligned}
\frac{1}{2}\left(\vec{v}_7 + \vec{v}_8 + \vec{v}_9\right) &= \left\langle \frac{1}{4}\left(-\sqrt{3} - 3\right), \frac{1}{4}\left(-\sqrt{3} - 1\right) \right\rangle \\
&= \frac{1}{2}\left(\sqrt{3} + 1\right)\left\langle -\frac{\sqrt{3}}{2}, -\frac{1}{2} \right\rangle \\
&= \frac{1}{2}\left(\sqrt{3} + 1\right)\vec{v}_8
\end{aligned}$$

15.)

$$\begin{aligned}
\overrightarrow{OP} &= \langle 1, 1, 0 \rangle \\
\overrightarrow{OQ} &= \langle 1, 2, -2 \rangle \\
\theta &= \arccos\left(\frac{\overrightarrow{OP} \cdot \overrightarrow{OQ}}{|\overrightarrow{OP}||\overrightarrow{OQ}|}\right) \\
&= \arccos\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4}
\end{aligned}$$

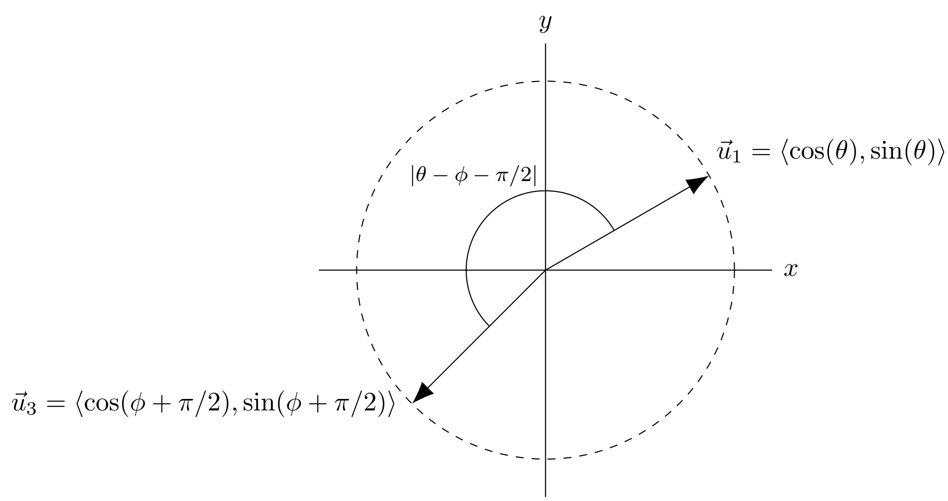
16.)



(a) Since $|\theta - \phi|$ is the angle between the unit vectors $\vec{u}_1$ and $\vec{u}_2$ and cosine is an even function:

$$\begin{aligned}\cos(|\theta - \phi|) &= \cos(\theta - \phi) \\ &= \frac{\vec{u}_1 \cdot \vec{u}_2}{|\vec{u}_1||\vec{u}_2|} \\ &= \cos(\theta)\cos(\phi) + \sin(\theta)\sin(\phi).\end{aligned}$$

Notice that we have assumed $|\theta - \phi|$ is no bigger than π (since we defined the angle between vectors to be the smaller of the two angles formed by the vectors). If this is not the case, then the angle between the two vectors would actually be $2\pi - |\theta - \phi|$. However, $\cos(2\pi - x) = \cos(x)$ and our answer would be no different!



(b) We need the following basic identities.

$$\begin{aligned}
\cos\left(\frac{\pi}{2} - \theta\right) &= \sin(\theta) \\
\sin\left(\frac{\pi}{2} - \theta\right) &= \cos(\theta) \\
\cos(-\theta) &= \cos(\theta) \\
\sin(-\theta) &= -\sin(\theta)
\end{aligned}$$

First, we note that

$$\begin{aligned}
\cos\left(\phi + \frac{\pi}{2}\right) &= \cos\left(\frac{\pi}{2} - (-\phi)\right) \\
&= \sin(-\phi) = -\sin(\phi), \\
\sin\left(\phi + \frac{\pi}{2}\right) &= \sin\left(\frac{\pi}{2} - (-\phi)\right) \\
&= \cos(-\phi) = \cos(\phi), \\
\cos\left(\left|\theta - \phi - \frac{\pi}{2}\right|\right) &= \cos\left(\theta - \phi - \frac{\pi}{2}\right) \\
&= \cos\left(\frac{\pi}{2} - (\theta - \phi)\right) = \sin(\theta - \phi).
\end{aligned}$$

This means that $\vec{u}_3 = \langle -\sin(\phi), \cos(\phi) \rangle$, and

$$\begin{aligned}
\sin(\theta - \phi) &= \frac{\vec{u}_1 \cdot \vec{u}_3}{|\vec{u}_1||\vec{u}_3|} \\
&= -\cos(\theta)\sin(\phi) + \sin(\theta)\cos(\phi).
\end{aligned}$$

Our comments about using $2\pi - |\theta - \phi - \pi/2|$ in case that angle is the smaller still holds here.

31.) Since these vectors are part of a circle, note that $\vec{A}$ and $\vec{B}$ are radii (and so have the same length). Clearly $\vec{B}+\vec{D}=\vec{A}$ by the usual parallelogram law of vector addition. This means $\vec{D}=\vec{A}-\vec{B}$. We also have $-\vec{A}+\vec{C}=\vec{B}$ which gives us $\vec{C}=\vec{A}+\vec{B}$.

$$\begin{aligned}\vec{C}\cdot\vec{D} &= \left(\vec{A}+\vec{B}\right)\cdot\left(\vec{A}-\vec{B}\right)\\ &= \vec{A}\cdot\vec{A}-\vec{A}\cdot\vec{B}+\vec{B}\cdot\vec{A}-\vec{B}\cdot\vec{B}\\ &= |\vec{A}|^2-|\vec{B}|^2\\ &= 0\end{aligned}$$

So, vectors $\vec{C}$ and $\vec{D}$ must be perpendicular.

36.) First, note that

$$\begin{aligned}|\vec{A}+\vec{B}|^2 &= \left(\vec{A}+\vec{B}\right)\cdot\left(\vec{A}+\vec{B}\right)\\ &= \vec{A}\cdot\vec{A}+\vec{A}\cdot\vec{B}+\vec{B}\cdot\vec{A}+\vec{B}\cdot\vec{B}\\ &= |\vec{A}|^2+2\vec{A}\cdot\vec{B}+|\vec{B}|^2.\end{aligned}$$

So, if $|\vec{A}+\vec{B}|^2=|\vec{A}|^2+|\vec{B}|^2$, then we must have $\vec{A}\cdot\vec{B}=0$ by the calculation above. But this means that $\vec{A}$ and $\vec{B}$ are perpendicular.

43.)

(a) **FALSE**

By our computation in the previous problem

$$|\vec{V}+\vec{W}|^2=|\vec{V}|^2+2\vec{V}\cdot\vec{W}+|\vec{W}|^2.$$

This means $|\vec{V}+\vec{W}|^2>|\vec{V}|^2+|\vec{W}|^2$ whenever $\vec{V}\cdot\vec{W}>0$. Geometrically, this happens when the angle between $\vec{V}$ and $\vec{W}$ is acute.

(b) **TRUE**

Since a straight line is the shortest distance between two points, then the sum of the lengths of any two sides of a “real” triangle must be greater than the length of the other side.

(c) **TRUE**

This is a basic property of the dot product.

(d) **FALSE**

In three-dimensions, the set of vectors perpendicular to a given vector form a plane (not a line).

46.)

(a) We can simply take the unit vector

$$\vec{V} = \langle \cos(45°), \sin(45°) \rangle = \frac{\sqrt{2}}{2} \langle 1, 1 \rangle.$$

(b) Clearly, we need to move to three-dimensions to find such a vector. Let $\vec{V} = \langle a, b, c \rangle$. To make things easy, we will assume that $\vec{V}$ has unit length (so, $a^2 + b^2 + c^2 = 1$). We have two other requirements:

$$\begin{aligned} \vec{V} \cdot \hat{\imath} &= a = \cos(60°) = \frac{1}{2}, \\ \vec{V} \cdot \hat{\jmath} &= b = \cos(60°) = \frac{1}{2}. \end{aligned}$$

This means that

$$\begin{aligned} \frac{1}{4} + \frac{1}{4} + c^2 &= 1 \\ c^2 &= \frac{1}{2} \\ c &= \pm \frac{\sqrt{2}}{2}. \end{aligned}$$

So, we can take

$$\vec{V} = \left\langle \frac{1}{2}, \frac{1}{2}, \frac{\sqrt{2}}{2} \right\rangle.$$

(c) If we want to make an angle θ with both $\hat{\imath}$ and $\hat{\jmath}$, then following the reasoning above, we must have $\vec{V} = \langle a, a, c \rangle$ with

$$\begin{aligned} \vec{V} \cdot \hat{\imath} = \vec{V} \cdot \hat{\jmath} &= a = \cos(\theta), \\ 2a^2 + c^2 &= 1. \end{aligned}$$

This means $c^2 = 1 - 2\cos^2(\theta)$. As such, we must have

$$\begin{aligned} 1 - 2\cos^2(\theta) &\geq 0 \\ \cos(\theta)^2 &\leq \frac{1}{2} \\ -\frac{\sqrt{2}}{2} \leq \cos(\theta) &\leq \frac{\sqrt{2}}{2} \\ 45° \leq \theta &\leq 135° \end{aligned}$$

Since $30°$ is outside this range, it is impossible for a vector to make an angle of $30°$ with both $\hat{\imath}$ and $\hat{\jmath}$.