Final Exam Formula Sheet

- For a regular curve $\vec{r}(t)$,
  - $\vec{T}(t) = \dfrac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|}$
  - $\vec{N}(t) = \dfrac{\vec{T}\,'(t)}{|\vec{T}\,'(t)|}$
  - $\vec{B}(t) = \vec{T}(t) \times \vec{N}(t)$
- For a regular curve $\vec{r}(t)$, the curvature is given by

$$\kappa(t) = \frac{|\vec{T}\,'(t)|}{|\vec{r}\,'(t)|} = \frac{|\vec{r}\,'(t) \times \vec{r}\,''(t)|}{|\vec{r}\,'(t)|^3}.$$

- **The Second Derivative Test for Functions of Two Variables**
  Left $f(x,y)$ be a twice differentiable function, and assume its second partial derivatives are continuous. Let (a,b) be a critical point for f, and define the Hessian of f at (a,b) to be

$$D(a,b) = f_{xx}(a,b)f_{yy}(a,b) - \left[f_{xy}(a,b)\right]^2.$$

  - If $D(a,b) < 0$, then (a,b) is a Saddle Point.
  - If $D(a,b) > 0$ and $f_{xx}(a,b) > 0$, then (a,b) is a Local Minimum.
  - If $D(a,b) > 0$ and $f_{xx}(a,b) < 0$, then (a,b) is a Local Maximum.
  - If $D(a,b) = 0$, the test is inconclusive.
- **Lagrange Multipliers**
  If a differentiable function f has a local maximum or local minimum on a constraint curve of the form $g =$constant (where g is a differentiable function), then there is a constant λ so that

$$\nabla f = \lambda \nabla g$$

  at the local extremum (so long as ∇g is not zero at the point in question).
- **Polar Coordinates**
  - $x = r\cos\theta, \quad y = r\sin\theta$
  - $r = \sqrt{x^2+y^2}, \quad \tan\theta = \frac{y}{x}$
  - $dA = r\; dr d\theta$
- **Cylindrical Coordinates**
  - $x = r\cos\theta, \quad y = r\sin\theta, \quad z = z$
  - $r = \sqrt{x^2+y^2}, \quad \tan\theta = \frac{y}{x}, \quad z = z$
  - $dV = r\; dr d\theta dz$

- **Spherical Coordinates**
  - $x = \rho \cos\theta \sin\varphi, \quad y = \rho \sin\theta \sin\varphi, \quad z = \rho \cos\varphi$
  - $\rho = \sqrt{x^2 + y^2 + z^2}, \quad \tan\theta = \frac{y}{x}, \quad \varphi = \arccos(z/\rho)$
  - $dV = \rho^2 \sin\varphi \, d\rho d\theta d\varphi$

- **Change of Variables in Multiple Integrals**
  Suppose a transformation T given by $x = g(u, v)$ and $y = h(u, v)$ is one-to-one (except perhaps on the boundary of $\mathcal{D}$), maps a region $\mathcal{D}$ in the uv-plane to a region $\mathcal{C}$ in the xy-plane, and g and h have continuous partial derivatives in a region containing $\mathcal{D}$. If f is a continuous function over the region $\mathcal{C}$, then

  $$\iint_{\mathcal{C}} f(x, y) dx\, dy = \iint_{\mathcal{D}} f\left(g(u, v), h(u, v)\right) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du\, dv,$$

  where the Jacobian of the transformation is given by

  $$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix}.$$

  The analogous statement holds for triple integrals.

- **Conservative Vector Fields and Path Independence**
  If a vector field $\vec{F}$ is defined on a simply connected open domain and the components of $\vec{F}$ have continuous partial derivatives, then $\vec{F}$ is conservative if and only if

  $$\nabla \times \vec{F} = \vec{0}$$

  (i.e. there is a potential function f so that $\nabla f = \vec{F}$). Line integrals of conservative vector fields are independent of path.

- **Green's Theorem**
  Suppose a vector field $\vec{F}(x, y) = P(x, y)\hat{\imath} + Q(x, y)\hat{\jmath}$ has continuous partial derivatives on an open set in the plane containing the region $\mathcal{D}$ whose boundary is a positively oriented, piecewise smooth simple closed curve $\mathcal{C}$. Then the counter-clockwise circulation around $\mathcal{C}$ is given by

  $$\oint_{\mathcal{C}} \vec{F} \cdot d\vec{r} = \oint_{\mathcal{C}} P dx + Q dy = \iint_{\mathcal{D}} \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \, dx\, dy.$$

- **Surface Integrals**
  Suppose a surface $\mathcal{S}$ is parametrized by $\vec{r}(u,v) = \langle x(u,v), y(u,v), z(u,v)\rangle$ over a region $\mathcal{D}$ in the uv-plane. Assuming the partial derivatives exist, the integral of a scalar function f over the surface $\mathcal{S}$ is computed by

  $$\iint_{\mathcal{S}} f\, dS = \iint_{\mathcal{D}} f(\vec{r}(u,v))|\vec{r}_u \times \vec{r}_v|\, du\, dv.$$

  Moreover, if $\mathcal{S}$ is orientable and the orientation is chosen appropriately, the flux of a vector field $\vec{F}$ through the surface is computed by

  $$\iint_{\mathcal{S}} \vec{F} \cdot d\vec{S} = \iint_{\mathcal{D}} \vec{F}(\vec{r}(u,v)) \cdot (\vec{r}_u \times \vec{r}_v)\, du\, dv.$$

- **Stokes' Theorem**
  Let $\mathcal{S}$ be a piecewise smooth, oriented surface that is bounded by a simple, closed, piecewise smooth curve $\mathcal{C} = \partial\mathcal{S}$ given the induced orientation from $\mathcal{S}$. Suppose a vector field $\vec{F}$ has continuous partial derivatives on an open region containing $\mathcal{S}$. Then

  $$\oint_{\mathcal{C}} \vec{F} \cdot d\vec{r} = \iint_{\mathcal{S}} \left(\nabla \times \vec{F}\right) \cdot d\vec{S}.$$

- **The Divergence Theorem**
  Let $\mathcal{W}$ be a simple solid region in space that is bounded by a closed surface $\mathcal{S} = \partial\mathcal{W}$ given the outward orientation. Suppose a vector field $\vec{F}$ has continuous partial derivatives on an open region containing $\mathcal{W}$. Then

  $$\iint_{\mathcal{S}} \vec{F} \cdot d\vec{S} = \iiint_{\mathcal{W}} \left(\nabla \cdot \vec{F}\right)\, dV.$$