Exam 5 Outline

Vector Calculus I

A. Line Integrals of Scalar and Vector-Valued Functions in 2 and 3 Dimensions

B. Path Independence, Conservative Vector Fields, and Potential Functions

C. Parametric Surfaces and Surface Integrals of Scalar and Vector-Valued Functions

Exam 5 Formula Sheet

- For a regular curve $\vec{r}(t)$,
  - $\vec{T}(t) = \dfrac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|}$
  - $\vec{N}(t) = \dfrac{\vec{T}\,'(t)}{|\vec{T}\,'(t)|}$
  - $\vec{B}(t) = \vec{T}(t) \times \vec{N}(t)$
- For a regular curve $\vec{r}(t)$, the curvature is given by

$$\kappa(t) = \frac{|\vec{T}\,'(t)|}{|\vec{r}\,'(t)|} = \frac{|\vec{r}\,'(t) \times \vec{r}\,''(t)|}{|\vec{r}\,'(t)|^3}.$$

- **The Second Derivative Test for Functions of Two Variables**
  Left $f(x,y)$ be a twice differentiable function, and assume its second partial derivatives are continuous. Let (a,b) be a critical point for f, and define the Hessian of f at (a,b) to be

$$D(a,b) = f_{xx}(a,b)f_{yy}(a,b) - \left[f_{xy}(a,b)\right]^2.$$

  - If $D(a,b) < 0$, then (a,b) is a Saddle Point.
  - If $D(a,b) > 0$ and $f_{xx}(a,b) > 0$, then (a,b) is a Local Minimum.
  - If $D(a,b) > 0$ and $f_{xx}(a,b) < 0$, then (a,b) is a Local Maximum.
  - If $D(a,b) = 0$, the test is inconclusive.
- **Lagrange Multipliers**
  If a differentiable function f has a local maximum or local minimum on a constraint curve of the form $g =$constant (where g is a differentiable function), then there is a constant λ so that

$$\nabla f = \lambda \nabla g$$

  at the local extremum (so long as ∇g is not zero at the point in question).
- **Polar Coordinates**
  - $x = r\cos\theta, \;\; y = r\sin\theta$
  - $r = \sqrt{x^2+y^2}, \;\; \tan\theta = \frac{y}{x}$
  - $dA = r\; dr d\theta$
- **Cylindrical Coordinates**
  - $x = r\cos\theta, \;\; y = r\sin\theta, \;\; z = z$
  - $r = \sqrt{x^2+y^2}, \;\; \tan\theta = \frac{y}{x}, \;\; z = z$
  - $dV = r\; dr d\theta dz$

- **Spherical Coordinates**
  - $x = \rho\cos\theta\sin\varphi, \;\; y = \rho\sin\theta\sin\varphi, \;\; z = \rho\cos\varphi$
  - $\rho = \sqrt{x^2+y^2+z^2}, \;\; \tan\theta = \frac{y}{x}, \;\; \varphi = \arccos(z/\rho)$
  - $dV = \rho^2 \sin\varphi \, d\rho d\theta d\varphi$

- **Change of Variables in Multiple Integrals**
  Suppose a transformation T given by $x = g(u,v)$ and $y = h(u,v)$ is one-to-one (except perhaps on the boundary of $\mathcal{D}$), maps a region $\mathcal{D}$ in the uv-plane to a region $\mathcal{C}$ in the xy-plane, and g and h have continuous partial derivatives in a region containing $\mathcal{D}$. If f is a continuous function over the region $\mathcal{C}$, then

  $$\iint_{\mathcal{C}} f(x,y)dx\,dy = \iint_{\mathcal{D}} f\left(g(u,v),h(u,v)\right)\left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv,$$

  where the Jacobian of the transformation is given by

  $$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix}.$$

  The analogous statement holds for triple integrals.

- **Conservative Vector Fields and Path Independence**
  If a vector field $\vec{F} = \langle P, Q, R\rangle$ is defined on a simply connected open domain and the components of $\vec{F}$ have continuous partial derivatives, then there is a potential function f so that $\nabla f = \vec{F}$ if and only if

  $$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}, \qquad \frac{\partial P}{\partial z} = \frac{\partial R}{\partial x}, \qquad \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}.$$

  Line integrals of conservative vector fields are independent of path.

- **Surface Integrals**
  Suppose a surface $\mathcal{S}$ is parametrized by $\vec{r}(u,v) = \langle x(u,v), y(u,v), z(u,v)\rangle$ over a region $\mathcal{D}$ in the uv-plane. Assuming the partial derivatives exist, the integral of a scalar function f over the surface $\mathcal{S}$ is computed by

  $$\iint_{\mathcal{S}} f\;dS = \iint_{\mathcal{D}} f(\vec{r}(u,v))|\vec{r}_u \times \vec{r}_v|\;du\,dv.$$

  Moreover, if $\mathcal{S}$ is orientable and the orientation is chosen appropriately, the flux of a vector field $\vec{F}$ through the surface is computed by

  $$\iint_{\mathcal{S}} \vec{F}\cdot d\vec{S} = \iint_{\mathcal{D}} \vec{F}(\vec{r}(u,v))\cdot(\vec{r}_u \times \vec{r}_v)\;du\,dv.$$