

# Exam 5 Outline

## Vector Calculus I

- A. Centers of Mass and Centroids
- B. Line Integrals of Scalar and Vector-Valued Functions in 2 and 3 Dimensions
- C. Path Independence, Conservative Vector Fields, and Potential Functions
- D. Green's Theorem (both circulation and flux forms)

# Exam 5 Formula Sheet

- For a regular curve  $\vec{r}(t)$ ,

$$- \vec{T}(t) = \frac{\vec{r}'(t)}{|\vec{r}'(t)|}$$

$$- \vec{N}(t) = \frac{\vec{T}'(t)}{|\vec{T}'(t)|}$$

$$- \vec{B}(t) = \vec{T}(t) \times \vec{N}(t)$$

- For a regular curve  $\vec{r}(t)$ , the curvature is given by

$$\kappa(t) = \frac{|\vec{T}'(t)|}{|\vec{r}'(t)|} = \frac{|\vec{r}'(t) \times \vec{r}''(t)|}{|\vec{r}'(t)|^3}.$$

- **The Second Derivative Test for Functions of Two Variables**

Let  $f(x, y)$  be a twice differentiable function, and assume its second partial derivatives are continuous. Let  $(a, b)$  be a critical point for  $f$ , and define the Hessian of  $f$  at  $(a, b)$  to be

$$D(a, b) = f_{xx}(a, b)f_{yy}(a, b) - [f_{xy}(a, b)]^2.$$

- If  $D(a, b) < 0$ , then  $(a, b)$  is a Saddle Point.
- If  $D(a, b) > 0$  and  $f_{xx}(a, b) > 0$ , then  $(a, b)$  is a Local Minimum.
- If  $D(a, b) > 0$  and  $f_{xx}(a, b) < 0$ , then  $(a, b)$  is a Local Maximum.
- If  $D(a, b) = 0$ , the test is inconclusive.

- **Lagrange Multipliers**

If a differentiable function  $f$  has a local maximum or local minimum on a constraint curve of the form  $g = \text{constant}$  (where  $g$  is a differentiable function), then there is a constant  $\lambda$  so that

$$\nabla f = \lambda \nabla g$$

at the local extremum (so long as  $\nabla g$  is not zero at the point in question).

- **Polar Coordinates**

- $x = r \cos \theta, \quad y = r \sin \theta$
- $r = \sqrt{x^2 + y^2}, \quad \tan \theta = \frac{y}{x}$
- $dA = r \, dr \, d\theta$

- **Cylindrical Coordinates**

- $x = r \cos \theta, \quad y = r \sin \theta, \quad z = z$
- $r = \sqrt{x^2 + y^2}, \quad \tan \theta = \frac{y}{x}, \quad z = z$
- $dV = r \, dr \, d\theta \, dz$

- **Spherical Coordinates**

- $x = \rho \cos \theta \sin \varphi, \quad y = \rho \sin \theta \sin \varphi, \quad z = \rho \cos \varphi$
- $\rho = \sqrt{x^2 + y^2 + z^2}, \quad \tan \theta = \frac{y}{x}, \quad \varphi = \arccos(z/\rho)$
- $dV = \rho^2 \sin \varphi \, d\rho d\theta d\varphi$

- **Change of Variables in Multiple Integrals**

Suppose a transformation  $T$  given by  $x = g(u, v)$  and  $y = h(u, v)$  is one-to-one (except perhaps on the boundary of  $\mathcal{D}$ ), maps a region  $\mathcal{D}$  in the  $uv$ -plane to a region  $\mathcal{C}$  in the  $xy$ -plane, and  $g$  and  $h$  have continuous partial derivatives in a region containing  $\mathcal{D}$ . If  $f$  is a continuous function over the region  $\mathcal{C}$ , then

$$\iint_{\mathcal{C}} f(x, y) dx dy = \iint_{\mathcal{D}} f(g(u, v), h(u, v)) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du dv,$$

where the Jacobian of the transformation is given by

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix}.$$

The analogous statement holds for triple integrals.

- **Conservative Vector Fields and Path Independence**

If a vector field  $\vec{F}$  is defined on a simply connected open domain and the components of  $\vec{F}$  have continuous partial derivatives, then  $\vec{F}$  is conservative if and only if

$$\nabla \times \vec{F} = \vec{0}$$

(i.e. there is a potential function  $f$  so that  $\nabla f = \vec{F}$ ). Line integrals of conservative vector fields are independent of path.

- **Green's Theorem**

Suppose a vector field  $\vec{F}(x, y) = P(x, y)\hat{i} + Q(x, y)\hat{j}$  has continuous partial derivatives on an open set in the plane containing the region  $\mathcal{D}$  whose boundary is a positively oriented, piecewise smooth simple closed curve  $\mathcal{C}$ . Then the counter-clockwise circulation around  $\mathcal{C}$  is given by

$$\oint_{\mathcal{C}} \vec{F} \cdot d\vec{r} = \oint_{\mathcal{C}} P dx + Q dy = \iint_{\mathcal{D}} \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} dx dy.$$

For the outward flux from  $\mathcal{C}$  (again, oriented positively),

$$\oint_{\mathcal{C}} \vec{F} \cdot \hat{n} ds = \oint_{\mathcal{C}} P dy - Q dx = \iint_{\mathcal{D}} \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} dx dy.$$